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Pierre Henry-Labordere

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COUNTERPARTY RISK VALUATION: A MARKED BRANCHING DIFFUSION APPROACH

PIERRE HENRY-LABORDÈRE

Abstract. The purpose of this paper is to design an algorithm for the computation of the counterparty risk which is competitive in regards of a brute force "Monte-Carlo of Monte-Carlo" method (with nested simulations). This is achieved using marked branching diffusions describing a Galton-Watson random tree. Such an algorithm leads at the same time to a computation of the (bilateral) counterparty risk when we use the default-risky or counterparty-riskless option values as mark-to-market. Our method is illustrated by various numerical examples.

1. Introduction

The recent financial crisis has highlighted the importance of credit valuation adjustment when pricing derivative contracts. Bilateral counterparty risk is the risk that the issuer of a derivative contract or the counterparty may default prior to the expiry and fail to make future payments. This market imperfection leads naturally for Markovian models to non-linear second-order parabolic partial differential equations (PDEs). More precisely, the non-linearity in the pricing equation affects none of the differential terms and depends on the positive part of the mark-to-market value of the derivative upon default. We have a so-called semi-linear PDE. The numerical solution of this equation is a formidable task that has attracted little attention from practitioners. For multi-asset portfolios, these PDEs which suffer from the curse of dimensionality cannot be solved with finite-difference schemes. We must rely on probabilistic methods. Up to now, it seems that a brute force intensive "Monte-Carlo of Monte-Carlo" method (with nested simulations) is the only tool available for this task.

In this paper, we rely on new advanced non-linear Monte-Carlo methods for solving these semi-linear PDEs. A first approach is to use the so-called first-order backward stochastic differential equations. Unfortunately, in practise this method requires the computation of conditional expectations using regressions. Finding good quality regressors is notably difficult, especially for multi-asset portfolios. This leads us to introduce a new method based on branching diffusions describing a marked Galton-Watson random tree. A similar algorithm can also be applied to obtain stochastic representations for solutions of a large class of semi-linear parabolic PDEs in which the non-linearity can be approximated by a polynomial function.

2. Credit Valuation Adjustment

2.1. **Semi-linear PDEs.** For completeness, we derive the PDE arising in counterparty risk valuation of a European derivative with a payoff ψ at maturity T. In short, depending on the (modeling)

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choice of the mark-to-market value of the derivative upon default, we will get two types of semi-linear PDEs that can be schematically written as

$$\partial_t u + \mathcal{L}u + r_0 u + r_1 u^+ = 0, \quad u(T,x) = \psi(x) \tag{1}$$

and

$$\begin{aligned} \partial_t u + \mathcal{L}u + r_0 u + r_1 M + r_2 M^+ &= 0, \quad u(T,x) = \psi(x) \\ \partial_t M + \mathcal{L}M + r_4 M &= 0, \quad M(T,x) = \psi(x) \end{aligned} \tag{2}$$

$\mathcal{L}$ is the Itô generator of a multi-dimensional diffusion process and r_i are arbitrary functions of t and x.

2.2. **PDE derivation.** We assume the issuer is allowed to dynamically trade d underlying assets $X_. \in \mathbb{R}_+^d$. Additionally, in order to hedge his credit risk on the counterparty name, he can trade a default risky bond, denoted P_t^2. Furthermore, the values of the underlyings are not altered by the counterparty default which is modeled by a Poisson jump process. For the sake of simplicity, we consider a constant intensity. This assumption can be easily relaxed, in particular the intensity can follow an Itô diffusion. For use below, expressions with a subscript 2 denote counterparty quantities. We consider the case of a long position in a single derivative whose value we denote u. In practice netting agreements apply to the global mark-to-market value of a pool of derivative positions - u would then denote the aggregate value of these derivatives. The processes X_t, P_t^2 satisfy under the risk-neutral measure $\mathbb{P}$ (we assume the market model is complete)

$$\begin{aligned} \frac{dX_t}{X_t} &= r dt + \sigma(t, X_t).dW_t \\ \frac{dP_t^2}{P_t^2} &= (r + \lambda_2) dt - dJ_t^2 \end{aligned}$$

with W_t a d-dimensional Brownian motion, J_t^2 a jump Poisson process with intensity λ_2 and r the interest rate. The no-arbitrage condition and the completeness of the market give that $e^{-rt}u(t, X_t)$ is a $\mathbb{P}$-martingale, characterized by

$$\partial_t u + \mathcal{L}u + \lambda_2 \left(\tilde{u} - u\right) - ru = 0$$

where $\mathcal{L}$ denotes the Itô generator of X and $\tilde{u}$ the derivative value after the counterparty has defaulted. At the default event, $\tilde{u}$ is given by[1]

$$\tilde{u} = RM^+ - M^-$$

with M the mark-to-market value of the derivative to be used in the unwinding of the position upon default and R the recovery rate. There is an ambiguity in the market about the convention for the mark-to-market value to be settled at default. There are two natural conventions (see [4] for discussions about the relevance of these conventions): The mark-to-market of the derivative is evaluated at the time of default with provision for counterparty risk or without.

1. Provision for counterparty risk, $M = u$:

$$\partial_t u + \mathcal{L}u - (1 - R)\lambda_2 u^+ - ru = 0, \quad u(T,x) = \psi(x) \tag{3}$$

In the particular case when the payoff $\psi(x)$ is negative, the solution is given by $e^{-r(T-t)}\mathbb{E}_{t,x}[\psi(X_T)]$.

2. No provision for counterparty risk:

$$\begin{aligned} &\partial_t u + \mathcal{L}u + \lambda_2 \left(RM^+ - M^- - u\right) - ru = 0, \quad u(T,x) = \psi(x) \\ &\partial_t M + \mathcal{L}M - rM = 0, \quad M(T,x) = \psi(x) \end{aligned} \tag{4}$$

[1] $X \equiv X^+ - X^-$.

In the case of collateralized positions, counterparty risk applies to the variation of the mark-to-market value of the corresponding positions experienced over the time it takes to qualify a failure to pay margin as a default event - typically a few days. In the latter case, the non-linearity u_t^+ should be substituted with $(u_t - u_{t+\Delta})^+$ where Δ is this delay. We will come back to this situation in the last section (see remark 5.1).

By proper discounting and replacing u by $-u$ for the sake of the presentation, these two PDEs can be cast into normal forms

$$\partial_t u + \mathcal{L}u + \beta\left(u^+ - u\right) = 0, \quad u(T,x) = \psi(x) : \text{PDE2} \tag{5}$$

$$\partial_t u + \mathcal{L}u + \frac{\beta}{1-R}\left((1-R)\mathbb{E}_{t,x}[\psi]^+ + R\mathbb{E}_{t,x}[\psi] - u\right) = 0, \quad u(T,x) = \psi(x) : \text{PDE1} \tag{6}$$

with $\beta \equiv \lambda_2(1-R) \in \mathbb{R}^+$. It is interesting to note that a similar semi-linear PDE type (5) appears also in the pricing of American options.

2.3. **American options.** The replication price of an American option with exercise payoff $\psi(x)$ satisfies a variational PDE:

$$\max\left(\partial_t u + \mathcal{L}u, \psi(x) - u\right) = 0, \quad u(T,x) = \psi(x)$$

This PDE can be converted into a semi-linear PDE (see [2] for details):

$$\partial_t u + \mathcal{L}u = 1_{\psi(x) \geq u}\mathcal{L}\psi(x), \quad u(T,x) = \psi(x)$$

Stochastic representations of this equation lead to well-known early exercise premium formulas of American options. Our algorithm can also be applied to this non-linear PDE. It does not require regressions as in the well-known Longstaff-Schwartz method [10] or a "Monte-Carlo of Monte-Carlo method" as in Rogers's dual algorithm [1, 14].

In the next section, we briefly list (non-linear) Monte-Carlo algorithms which can be used to solve PDEs (5)-(6) and highlight their weaknesses in the context of credit valuation adjustment.

3. Non-linear Monte-Carlo algorithms

3.1. **A brute force algorithm.** Using Feynman-Kac's formula, the solution of PDE (5) can be represented stochastically as

$$u(t,x) = e^{-\beta(T-t)}\mathbb{E}_{t,x}[\psi(X_T)] + \int_t^T \beta e^{-\beta(s-t)}\mathbb{E}_{t,x}[u^+(s,X_s)]ds \tag{7}$$

with X an Itô diffusion with generator $\mathcal{L}$ and $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot|X_t = x]$. By assuming that the intensity β is small, we get the approximation (this is exact for PDE (6)[2])

$$u(t,x) = e^{-\beta(T-t)}\mathbb{E}_{t,x}[\psi(X_T)] + \beta e^{-\beta(T-t)}\int_t^T \mathbb{E}_{t,x}[(\mathbb{E}_{s,X_s}[\psi(X_T)])^+]ds + O(\beta^2) \tag{8}$$

Then, at a next step, we discretise the Riemann integral

$$u(t,x) \simeq e^{-\beta(T-t)}\mathbb{E}_{t,x}[\psi(X_T)] + \beta e^{-\beta(T-t)}\sum_{i=1}^{n}\mathbb{E}_{t,x}[\left(\mathbb{E}_{t_i,X_{t_i}}[\psi(X_T)]\right)^+]\Delta t_i$$

This last expression can be numerically tackled by using a brute force "Monte-Carlo of Monte-Carlo" method. The second MC is used to compute $\mathbb{E}_{t_i,X_{t_i}}\psi(X_T)]$ on each path generated by the first MC algorithm. Although straightforward, this method suffers from the curse of dimensionality

[2]Precisely, we get $e^{-\lambda_2(T-t)}\mathbb{E}_{t,x}[\psi(X_T)] + \lambda_2\int_t^T e^{-\lambda_2(s-t)}\mathbb{E}_{t,x}[(1-R)\left(\mathbb{E}_{s,X_s}[\psi(X_T)]\right)^+ + R\mathbb{E}_{s,X_s}[\psi(X_T)]]ds$.

and requires generating $O(N_1 \times N_2)$ paths. Due to this complexity, the literature focuses on exposition of linear portfolios for which the second MC can be skipped by using closed-form formulas or low-dimensional parametric regressions (see for example [3] in which the authors consider the pricing of CMS spread option and CCDSs).

Could we design a simple (non-linear) Monte-Carlo algorithm which solves our PDEs (5)-(6), without relying on an approximation such as (8)? This is the purpose of this paper.

3.2. **Backward stochastic differential equations.** A first approach is to simulate a backward stochastic differential equation (in short BSDE):

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t).dW_t, \quad X_0 = x \tag{9}$$

$$dY_t = -\beta Y_t^+ dt + Z_t \sigma(t, X_t).dW_t \tag{10}$$

$$Y_T = \psi(X_T) \tag{11}$$

where (Y, Z) are required to be adapted processes and $\mathcal{L} = \sum_i \mu_i \partial_{x^i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial^2_{x^i x^j}$. BSDEs differ from (forward) SDEs in that we impose the terminal value (see Equation (11)). Under the condition $\psi \in \mathrm{L}^2(\Omega)$, this BSDE admits a unique solution [13]. A straightforward application of Itô's lemma gives that the solution of this BSDE is $\left(Y_t = e^{\beta(T-t)}u(t, X_t), Z_t = e^{\beta(T-t)}\nabla_x u(t, X_t)\right)$ with u the solution of PDE (5). This leads to a Monte-Carlo like numerical solution of (5) via an efficient discretization scheme for the above BSDE.

This BSDE can be discretized by an Euler-like scheme ($Y_{t_{i-1}}$ is forced to be $\mathcal{F}_{t_{i-1}}$-adapted, $(\mathcal{F}_t)_{t\geq 0}$ being the natural filtration generated by the Brownian motions):

$$\mathbb{E}_{t_{i-1}}[Y_{t_i}] - Y_{t_{i-1}} = -\beta\Delta t_i \left(\theta Y^+_{t_{i-1}} + (1-\theta)\mathbb{E}_{t_{i-1}}[Y_{t_i}]^+\right)$$

with $\theta \in [0, 1]$. This is equivalent to (we take $\theta\beta\Delta t_i < 1$)

$$Y_{t_{i-1}} = \mathbb{E}_{t_{i-1}}[Y_{t_i}]\left(1_{\mathbb{E}^{\mathbb{P}}_{t_{i-1}}[Y_{t_i}]>0}\frac{1 + (1-\theta)\beta\Delta t_i}{1 - \theta\beta\Delta t_i} + 1_{\mathbb{E}_{t_{i-1}}[Y_{t_i}]<0}\right)$$

This requires the computation of the conditional expectation $\mathbb{E}^{\mathbb{P}}_{t_{i-1}}[Y_{t_i}]$ (in practise by regression methods) which could be quite difficult and time-consuming, especially for multi-asset portfolios.

3.3. **Gradient representation.** A more powerful approach is synthesized by the following proposition which relies on Kunita's stochastic flows of diffeomorphisms (see [16]). Let u be the solution of the *one-dimensional* semi-linear PDE

$$\partial_t u + \frac{1}{2}\sigma^2(t, x)\partial_x^2 u + f(u) = 0 \tag{12}$$

with the terminal condition $u(T, x) = \psi(x)$. By differentiating equation (12) with respect to x (assuming smoothness of the coefficients) we get

$$\partial_t \Delta + \left((\sigma\partial_x\sigma)\,\partial_x + \frac{1}{2}\sigma^2(t, x)\partial_x^2\right)\Delta + f'(u)\,\Delta = 0 \tag{13}$$

with the terminal condition $\Delta(T, x) = \psi'(x)$. The equation satisfied by the gradient Δ is then interpreted as a (linear) Fokker-Planck PDE. We have the following representation [16]

$$u(t, x) = -\int_{\mathbb{R}^+}\psi'(a)da\ \mathbb{E}_t[1(X^a_T - x)e^{\int_t^T f'(u(T+t-s, X^a_s))ds}]$$

where the Itô process X^a_s is the solution to

$$dX^a_s = \sigma(T + t - s, X^a_s)dB_s + (\sigma\partial_x\sigma)\,(T + t - s, X^a_s)ds\ ,\ s \in [t, T]\ ,\ X^a_t = a$$

B_s is a standard Brownian. This representation leads to a particle algorithm [16]. Although appealing, this (forward) approach is only applicable in the one-dimension setup for which we can use a PDE solver. Can we design a similar forward algorithm applicable in higher dimensions? This leads us to branching diffusions.

3.4. **Branching diffusions: an introduction.** Branching diffusions have been first introduced by McKean [8] to give a probabilistic representation of the Kolmogorov-Petrovskii-Piskunov PDE and more generally of semi-linear PDEs of the type

$$\partial_t u + \mathcal{L}u + \beta(t)\left(\sum_{k=0}^{\infty} p_k u^k - u\right) = 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^d \tag{14}$$
$$u(T,x) = \psi(x) \quad \text{in} \quad \mathbb{R}^d$$

with $\beta(\cdot) \in \mathbb{R}^+$. Here the non-linearity is a power series in u where the coefficients satisfy the restrictive condition:

$$f(u) \equiv \sum_{k=0}^{\infty} p_k u^k, \quad \sum_{k=0}^{\infty} p_k = 1 \quad 0 \leq p_k \leq 1 \tag{15}$$

The probabilistic interpretation of such an equation goes as follows: Let a single particle start at the origin, perform an Itô diffusion on $\mathbb{R}^d$ with generator $\mathcal{L}$, after a mean $\beta(\cdot)$ exponential time (independent of X) die and produce k descendants with probability p_k ($k = 0$ means that the particle dies without generating descendants). Then, the descendants perform independent Itô diffusions on $\mathbb{R}^d$ (with same generator $\mathcal{L}$) from their birth locations, die and produce descendants after a mean $\beta(\cdot)$ exponential times, etc. This process is called a d-dimensional branching diffusion with a branching rate $\beta(\cdot)$. β can also depend spatially on x or be itself stochastic (Cox process). We note $Z_t \equiv \left(z_t^1, \ldots, z_t^{N_t}\right) \in \mathbb{R}^{d \times N_t}$ the locations of the particles alive at time t and N_t the number of particles at t (see Fig. 1 for examples with 2 and 3 descendants). We consider then the multiplicative functional defined by[3]

$$\hat{u}(t,x) = \mathbb{E}_{t,x}\Big[\prod_{i=1}^{N_T} \psi(z_T^i)\Big] \tag{16}$$

where $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot|N_t = 1, z_t^1 = x]$. Note that as N_T can become infinite when $m = \sum_{k=0}^{\infty} k p_k > 1$ (super-critical regime, see [12]), a sufficient condition on ψ in order to have a well-behaved product is $|\psi| < 1$. Then $\hat{u}$ solves the semi-linear PDE (14). This stochastic representation can be understood as follows: Mathematically, by conditioning on τ, the first-time to jump of a Poisson process with intensity $\beta(t)$, we get from (16)

$$\hat{u}(t,x) = \mathbb{E}_{t,x}[1_{\tau \geq T}\psi(z_T^1)] + \mathbb{E}_{t,x}[1_{\tau < T}\sum_{k=0}^{\infty} p_k \mathbb{E}_\tau[\prod_{j=1}^{k} \prod_{i=1}^{N_T^j(\tau)} \psi(z_T^{i,j,z_\tau})]$$

[3] $\prod^{N_T=0} \equiv 1$ by convention.

where z_T^{i,j,z_τ} is the position of the i-th particle at maturity T produced by the j-th particle generated at time τ. By using the independence and the strong Markov property, we obtain

$$\begin{aligned}
\hat{u}(t,x) &= \mathbb{E}_{t,x}[1_{\tau\geq T}\psi(z_T^1)] + \sum_{k=0}^{\infty}\mathbb{E}_{t,x}[1_{\tau<T}p_k\prod_{j=1}^{k}\mathbb{E}_\tau[\prod_{i=1}^{N_T^j(\tau)}\psi(z_T^{i,j,z_\tau})] \\
&= \mathbb{E}_{t,x}[1_{\tau\geq T}\psi(z_T^1)] + \mathbb{E}_{t,x}[1_{\tau<T}\sum_{k=0}^{\infty}p_k\prod_{j=1}^{k}\hat{u}(\tau,z_\tau^1)] \\
&= \mathbb{E}_{t,x}[1_{\tau\geq T}\psi(z_T^1)] + \sum_{k=0}^{\infty}p_k\mathbb{E}_{t,x}[\hat{u}^k(\tau,z_\tau^1)1_{\tau<T}] \\
&= \mathbb{E}_{t,x}[e^{-\int_t^T\beta(s)ds}\psi(z_T^1)] + \int_t^T\sum_{k=0}^{\infty}p_k\mathbb{E}_{t,x}[\beta(s)e^{-\int_t^s\beta(u)du}\hat{u}^k(s,z_s^1)]ds
\end{aligned}$$

Then, by assuming that $||\psi||_\infty < 1$, $\hat{u}$ is uniformly bounded by 1 in $[0,T]\times\mathbb{R}^d$ and we get from the Feynman-Kac formula that $\hat{u}$ is a viscosity solution to PDE (14) (see Theorem 6.4 in [17]). By assuming that PDE (14) satisfies a comparison principle, we conclude that $u=\hat{u}$.

A first attempt in order to obtain a larger class of non-linearities than those defined by (15) is to consider an infinite collection of branching diffusions, the so-called super-diffusions. (15) is then extended to

$$\Psi(u) = au + bu^2 + \int_0^\infty n(dr)[e^{-ru} - 1 + ru] \tag{17}$$

where $a\geq 0$, $b\geq 0$ and n is a Radon measure on $(0,\infty)$ satisfying $\int_0^\infty(r\wedge r^2)n(dr)<\infty$. The class of non-linearity as defined by (17) is more general than (15), in particular contains $au+bu^2$ with arbitrary positive coefficients a and b. Unfortunately, this requires a large number of branching diffusions (as the default intensity diverges) and the non-linearity is still restrictive. This leads us to introduce a new class of branching diffusions that can be traced back to Le Jan-Sznitman [9] in the context of stochastic (Fourier) representations of solutions of the incompressible Navier-Stokes equation.

4. Marked branching diffusions

The PDE (14) should be compared with the semi-linear PDE (5) arising in the pricing of counterparty risk. It seems too restrictive and unreasonable to approximate the non-linearity u^+ by a polynomial of type (15) or even (17). A natural question is therefore to search if this construction can be generalized for an arbitrary polynomial for which the PDE is

$$\partial_t u + \mathcal{L}u + \beta(F(u)-u) = 0 \tag{18}$$

with $F(u)=\sum_{k=0}^M a_k u^k$ an M-order polynomial in u that we write for convenience $F(u)=\sum_{k=0}^M\left(\frac{a_k}{p_k}\right)p_k u^k$. We will show below that this can be achieved by counting the branching of each monomial u^k.

Assumption (Comp): In order to have uniqueness in the viscosity sense, we assume PDE (18) satisfies a comparison principle for sub- and super-solutions (see [7]).

For each Galton-Watson tree, we denote $\omega_k\in\mathbb{N}$ the number of branching of monomial type u^k with $k\in\{0,\ldots,M\}$. The descendants are drawn with an arbitrary distribution p_k - for example we can take a uniform distribution $p_k=\frac{1}{M+1}$ (see an other choice in section 4.3). In Fig. 1, we

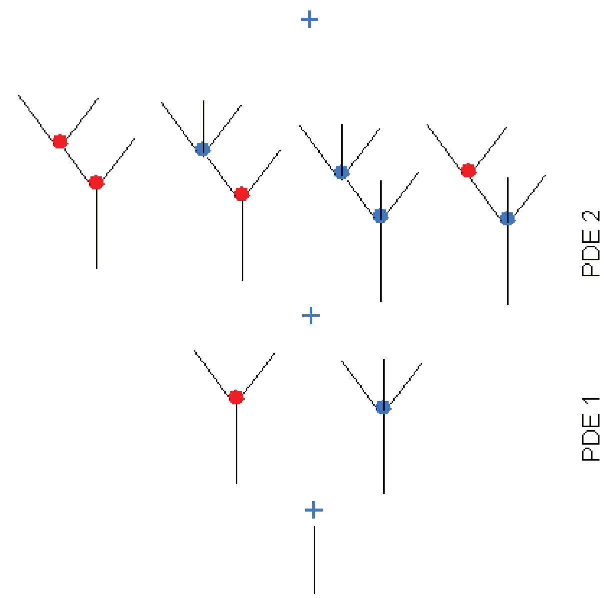

FIGURE 1. Marked Galton-Watson random tree for the non-linearity $F(u) = \frac{a}{p_2}p_2u^2 + \frac{b}{p_3}p_3u^3$. The red (resp. blue) vertex corresponds to the weight $\frac{a}{p_1}$ (resp. $\frac{b}{p_2}$). The diagram with two red vertices has the weights $(\omega_1 = 2, \omega_2 = 0)$.

have drawn the diagrams for the non-linearity $F(u) = \frac{a}{p_2}p_2u^2 + \frac{b}{p_3}p_3u^3$ up to two defaults. We then define the multiplicative functional:

Main formula:

$$\hat{u}(t,x) = \mathbb{E}_{t,x}\Big[\prod_{i=1}^{N_T}\psi(z_T^i)\prod_{k=0}^{M}\left(\frac{a_k}{p_k}\right)^{\omega_k}\Big]\ ,\ \omega_k = \sharp\text{branching type } k \tag{19}$$

We state our main result (the proof is reported in the appendix):

Theorem 4.1. *Let us assume that $\hat{u} \in \mathrm{L}^\infty([0,T]\times\mathbb{R}^d)$ and (**Comp**) holds. The function $\hat{u}(t,x)$ is the unique viscosity solution of (18).*

Diagrammar interpretation. From Feynman-Kac's formula, we have

$$u(t,x) = \mathbb{E}_{t,x}[1_{\tau\geq T}\psi(X_T)] + \mathbb{E}_{t,x}[F(u(\tau,X_\tau))1_{\tau<T}] \tag{20}$$

This integral equation can be recursively solved in terms of multiple exponential random times τ_i:

$$\begin{aligned} u(t,x) &= \mathbb{E}_{t,x}[1_{\tau_0\geq T}\psi(X_T)] \\ &+ \mathbb{E}_{t,x}[F\left(\mathbb{E}_{\tau_0}[1_{\tau_1\geq T}\psi(X_T)] + \mathbb{E}_{\tau_0}[F(\mathbb{E}_{\tau_1}[1_{\tau_2\geq T}\psi(X_T)])1_{\tau_1<T}]\right)1_{\tau_0<T}] + \cdots \end{aligned} \tag{21}$$

Each term can be interpreted as a Feynman diagram (see Fig. 1) representing the trajectory of a branching diffusion with a weight depending on the branching of each monomial. For example in Fig. 1, the diagram with two red vertices corresponds to

$$\left(\frac{a_2}{p_2}\right)^2\mathbb{E}_{t,x}[1_{\tau_0<T}\mathbb{E}_{\tau_0}[1_{\tau_1\geq T}\psi(X_T)]\mathbb{E}_{\tau_0}[1_{\tau_2<T}\mathbb{E}_{\tau_2}[1_{\tau_3\geq T}\psi(X_T)]^2]]$$

By assuming that the series (21) is convergent, one can guess that the solution is given by our multiplicative functional (19).

In the next section, we focus on convergence issues and deduce a sufficient condition to ensure that $\hat{u} \in \mathrm{L}^\infty([0,T] \times \mathbb{R}^d)$ if ψ is bounded.

4.1. **Convergence issues.** The number of particles $N(\omega)$, produced by the branching $\omega \equiv (\omega_0, \ldots, \omega_M)$, is

$$N(\omega) = \sum_{k=0}^{M} (k-1)\omega_k + 1 \tag{22}$$

The probability of such a configuration satisfies the recurrence equation

$$\begin{aligned} \mathbb{P}(T|\omega) = \beta \sum_{k=0}^{M} & \int_0^T dt \mathbb{P}(t|\omega_0, \ldots, \omega_k - 1, \ldots, \omega_M) N(\omega_0, \ldots, \omega_k - 1, \ldots, \omega_M) \\ & p_k e^{-\beta k(T-t)} e^{-\beta(T-t)(N(\omega_0,\ldots,\omega_k-1,\ldots,\omega_M)-1)} \end{aligned} \tag{23}$$

Indeed, if we have a tree with a branching $(\omega_0, \ldots, \omega_k - 1, \ldots, \omega_M)$ at time t, a particle among the $N(\omega_0, \ldots, \omega_k - 1, \ldots, \omega_M)$ particles must die and produce k descendants (with probability $p_k \beta e^{-k\beta(T-t)}$). The remaining $N(\omega_0, \ldots, \omega_k - 1, \ldots, \omega_M) - 1$ particles must survive until maturity T (with probability $e^{-\beta(T-t)(N(\omega_0,\ldots,\omega_k-1,\ldots,\omega_M)-1)}$).

We prove in the appendix that the Laplace transform of $\mathbb{P}$, $\hat{\mathbb{P}}(T,c) = \mathbb{E}[\prod_{k=0}^M e^{-c_k \omega_k}]$, satisfies the equation

$$\int_1^{\hat{\mathbb{P}}(T,c)} \frac{ds}{-s + \sum_{k=0}^M p_k e^{-c_k} s^k} = \beta T \ \text{ if } \sum_{k=0}^{M} p_k e^{-c_k} \neq 1 \tag{24}$$

$$\hat{\mathbb{P}}(T,c) = 1 \ \text{ if } \sum_{k=0}^{M} p_k e^{-c_k} = 1 \tag{25}$$

In the particular case of one branching type $k \neq 1$, we have

$$\hat{\mathbb{P}}(T, c_k) = \frac{e^{\frac{c_k}{k-1}}}{\left(1 - e^{\beta T(k-1)} + e^{c_k + \beta T(k-1)}\right)^{\frac{1}{k-1}}}$$

By assuming that $\psi \in \mathrm{L}^\infty(\mathbb{R}^d)$, the expectation in (19) can then be bounded by

$$|\hat{u}(0,x)| \quad \leq \quad \mathbb{E}_{0,x}[\prod_{k=0}^{M} \left(\frac{|a_k|}{p_k}\right)^{\omega_k} ||\psi||_\infty^{N(\omega)}] = ||\psi||_\infty \hat{\mathbb{P}}\left(T, -\ln\frac{|a_k|}{p_k} - \ln ||\psi||_\infty^{k-1}\right) \tag{26}$$

from which we deduce a sufficient condition for convergence:

Proposition 4.2. *Let us assume that $\psi \in \mathrm{L}^\infty(\mathbb{R}^d)$. Set $p(s) = \beta\left(-s + \sum_{k=0}^M |a_k| ||\psi||_\infty^{k-1} s^k\right)$.*

(1) *Case $\sum_{k=0}^M |a_k| ||\psi||_\infty^{k-1} > 1$: We have $\hat{u} \in \mathrm{L}^\infty([0,T] \times \mathbb{R}^d)$ (as defined by (19)) if there exists $X \in \mathbb{R}_+^*$ such that*

$$\int_1^X \frac{ds}{p(s)} = T$$

In the particular case of one branching type k, the sufficient condition for convergence reads as

$$|a_k| ||\psi||_\infty^{k-1} \left(1 - e^{-\beta T(k-1)}\right) < 1$$

(2) *Case* $\sum_{k=0}^{M} |a_k| ||\psi||_\infty^{k-1} \leq 1$*:* $\hat{u} \in \mathrm{L}^\infty([0,T] \times \mathbb{R}^d)$ *for all* T*.*

Note that our blow-up criteria does not depend on the probabilities p_k as expected.

4.2. **PDE (6).** We assume that the function $(1-R)x^+ + Rx$ can be well approximated by a polynomial $F(x)$ (see section 5) and we consider the PDE

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + \frac{\beta}{1-R}\left(F(\mathbb{E}_{t,x}[\psi(X_T)]) - u(t,x)\right) = 0, \quad u(T,x) = \psi(x)$$

From Feynman-Kac's formula, we have

$$u(t,x) = \mathbb{E}_{t,x}[1_{\tau \geq T}\psi(X_T)] + \mathbb{E}_{t,x}[F(\mathbb{E}_\tau[\psi(X_T)])1_{\tau<T}]$$

with τ a Poisson default time with intensity $\beta/(1-R)$. As compared to the previous section, we have the term $\mathbb{E}_{t,x}[F(\mathbb{E}_\tau[\psi(X_T)])1_{\tau<T}]$ instead of $\mathbb{E}_{t,x}[F(u(\tau, X_\tau))1_{\tau<T}]$. This term can be computed using the previous algorithm by imposing that the particle can default only once. This corresponds to the first three diagrams in Fig. (1). As N_T is valued in $[0,M]$, our formula (19) is convergent here for all polynomial non-linearities.

As a conclusion, without any modification, the branching particle algorithm can solve the two PDEs (5)-(6) modulo that the non-linearly u^+ can be fairly well approximated by a polynomial.

4.3. **Optimal probabilities p_k.** Is there a better choice than an uniform distribution $p_k = \frac{1}{M+1}$ for improving the convergence?

For the PDE (5), the variance of the algorithm (depending on the probabilities p_k) is bounded by (see Equation (26))

$$||\psi||_\infty \hat{\mathbb{P}}\left(T, -2\ln\frac{|a_k|}{p_k} - 2\ln||\psi||_\infty^{k-1}\right)$$

By minimizing with respect to p_k, we get

$$p_k = \frac{|a_k| ||\psi||_\infty^k}{\sum_{i=0}^{M} |a_i| ||\psi||_\infty^i} \tag{27}$$

Similarly, for the PDE (6), the variance (depending on the probabilities p_k) is bounded by

$$\sum_{k=0}^{M} \frac{a_k^2}{p_k} ||\psi||_\infty^{2k} \beta T e^{-\beta T}$$

By minimizing with respect to p_k, we get also (27).

We recall that the population in the Galton-Watson tree disappears in finite time almost surely if $m \equiv \sum_{k=0}^{M} k p_k \leq 1$ (see [12]). In the super-critical case $m > 1$, the population explodes at a finite time $T_{\exp}$ with probability $1 - s_0$ where $s_0 = \inf\{s \in [0,1], \sum_{k=0}^{M} p_k s^k = s\}$. From (27), we are in the super-critical case if $\sum_{k=0}^{M}(k-1)|a_k| ||\psi||_\infty^k > 0$.

4.4. **Numerical Experiments.** Before applying our algorithm to the problem of credit valuation adjustment, we check it on polynomials which do not belong to the classes defined by (15) and (17).

N	Fair(PDE2)	Stdev(PDE2)	Fair(PDE1)	Stdev(PDE1)
12	20.78	0.78	21.31	0.79
14	22.25	0.39	21.37	0.39
16	21.97	0.19	21.76	0.20
18	21.90	0.10	21.51	0.10
20	21.86	0.05	21.48	0.05
22	**21.81**	0.02	**21.50**	0.02

Table 1. MC price quoted in percent as a function of the number of MC paths 2^N. PDE pricer(PDE1) = **21.82**. PDE pricer(PDE2) = **21.50**. Non-linearity $F(u) = \frac{1}{2}\left(u^3 - u^2\right)$.

N	Fair(PDE2)	Stdev(PDE2)	Fair(PDE1)	Stdev(PDE1)
12	21.14	0.78	20.00	0.78
14	21.56	0.38	19.90	0.39
16	21.62	0.19	20.25	0.20
18	21.31	0.10	20.39	0.10
20	21.38	0.05	20.36	0.05
22	**21.36**	0.02	**20.40**	0.02

Table 2. MC price quoted in percent as a function of the number of MC paths 2^N. PDE pricer(PDE1) = **21.37**. PDE pricer(PDE2) = **20.39**. Non-linearity $F(u) = \frac{1}{3}\left(u^3 - u^2 - u^4\right)$.

4.4.1. *Experiment 1.* We have implemented our algorithm for the two PDE types

$$\partial_t u + \mathcal{L}u + \beta(F(u) - u) = 0, \quad u(T,x) = 1_{x>1} \ : \ \text{PDE2}$$

and

$$\partial_t u + \mathcal{L}u + \beta(F(\mathbb{E}_{t,x}[1_{X_T>1}]) - u) = 0, \quad u(T,x) = 1_{x>1} \ : \ \text{PDE1}$$

with $F(u) = \frac{1}{2}\left(u^3 - u^2\right)$. $\mathcal{L}$ is the Itô generator of a geometric Brownian motion with a volatility $\sigma_{\text{BS}} = 0.2$ and the Poisson intensity is $\beta = 0.05$. In financial terms, this corresponds to a CDS spread around 500 basis points. The maturity is $T = 10$ years. From (27), we note that our optimal probability distributions for PDE1 and PDE2 coincide with the uniform distribution. Moreover Proposition (4.2) gives that the solution does not blow up.

The numerical method has been checked against a one-dimensional PDE solver with a fully implicit scheme (see Table. 1) for which we find $u = 21.82\%$ (PDE1) and $u = 21.50\%$ (PDE2). Note that this algorithm converges as expected and the error is properly indicated by the Monte-Carlo standard deviation estimator (see column Stdev).

4.4.2. *Experiment 2.* Same test with $F(u) = \frac{1}{3}\left(u^3 - u^2 - u^4\right)$ (see Table. 2) and same comments as above.

4.4.3. *Experiment 3: Blow-up explosion.* It is well-known that the semi-linear PDE in $\mathbb{R}^d$

$$\partial_t u + \mathcal{L}u + u^2 = 0$$

blows up in finite time if $d \leq 2$ for any bounded positive payoff (see [15]). We deduce that the PDE with the non-linearity $F(u) = u^2 + u$ blows up in finite time ($T_{\max}$) in one dimension. Using

Maturity(Year)	BBM alg.(Stdev)	PDE
0.5	71.66(0.09)	71.50
1	157.35(0.49)	157.17
1.1	$\infty(\infty)$	∞

Table 3. MC price quoted in percent as a function of the maturity for the non-linearity $F(u) = u^2 + u$. $\psi(x) \equiv 1_{x>1}$.

Proposition (4.2), our sufficient condition reads as

$$T_{\max}||\psi||_\infty < 1$$

We have verified this explosion when the maturity T is greater than 1 year (in our case $\psi = 1_{x>0}$, $||\psi||_\infty = 1$) using our algorithm (and a PDE solver as a benchmark). Note that for $T = 1$, the algorithm starts to blow up (see Stdev = 0.49). A different stochastic representation can be obtained by setting $u = e^{(T-t)}v$. We get

$$\partial_t v + \mathcal{L}v + e^{(T-t)}v^2 - v = 0\ ,\ v(T,x) = \psi(x)$$

and this can be interpreted as a binary tree with a weight $e^{(T-\tau)}$. Our stochastic representation gives then

$$u(t,x) = e^{T-t}\mathbb{E}_{t,x}\Big[\prod_{i=1}^{N_T}\psi(z_T^i)e^{\sum_{i=1}^{\sharp\text{branching}}(T-\tau_i)}\Big] \tag{28}$$

where τ_i is the time where the i-th branching appears. This representation (28) appears in [11] and was used to reproduce Sugitani's blow-up criteria [15].

5. Credit valuation adjustment algorithm

In the previous section, we have assumed that the payoff was bounded: $\psi \in \mathrm{L}^\infty$. Then, the solution u can then be written as $v = \frac{u}{||\psi||_\infty}$ where v satisfies

$$\partial_t v + \mathcal{L}v + \beta\left(v^+ - v\right) = 0, \quad ||v(T,\cdot)|| \leq 1 \tag{29}$$

Therefore, by re-scaling, we can consider that the payoff satisfies the condition $||\psi||_\infty \leq 1$. The condition $\psi \in \mathrm{L}^\infty$ can be easily relaxed as observed in ([6], see Remark 3.7). Let ψ be a payoff with α-exponential growth for some $\alpha > 0$. We scale the solution by an arbitrary smooth positive function ρ given by

$$\begin{aligned}\rho(x) &\equiv e^{\alpha|x|} \text{ for } |x| \geq M\\ \tilde{v}(t,x) &\equiv \rho^{-1}(x)v(t,x)\end{aligned}$$

If we write the linear operator $\mathcal{L}$ as $\mathcal{L}v = \mu(t,x)\partial_x v + \frac{1}{2}\sigma^2(t,x)\partial_x^2 v$, then $\tilde{v}$ satisfies a PDE[4] with the same non-linearity βv^+:

$$\partial_t\tilde{v} + \tilde{\mathcal{L}}\tilde{v} + \beta\left(\tilde{v}^+ - \tilde{v}\right) = 0$$

with $\tilde{\mathcal{L}}\tilde{v} = \left(\mu + \sigma^2\rho^{-1}\partial_x\rho\right)\partial_x\tilde{v} + \frac{1}{2}\sigma^2(t,x)\partial_x^2\tilde{v} + \left(\mu\rho^{-1}\partial_x\rho + \frac{1}{2}\rho^{-1}\sigma^2\partial_x^2\rho\right)\tilde{v}$.

What remains to be done in order to use (19) is to approximate v^+ by a polynomial $F(v)$:

$$\partial_t v + \mathcal{L}v + \beta\left(F(v) - v\right) = 0, \quad v(T,x) = \psi(x) \tag{30}$$

[4] $\tilde{\mathcal{L}}$ is written in $d = 1$. A similar expression can be obtained in a multi-dimensional setup.

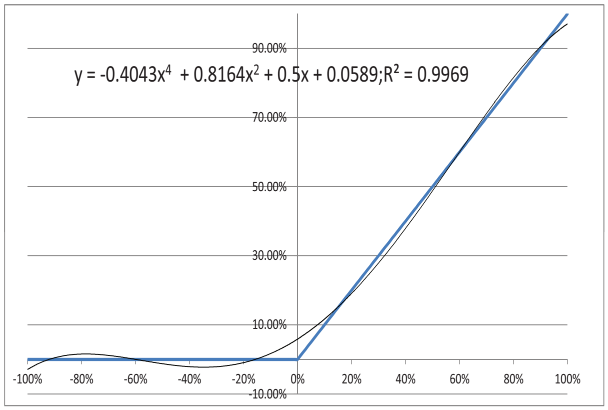

FIGURE 2. u^+ versus its polynomial approximation on $[-1,1]$.

In our numerical experiments, we take (see Fig. 2)

$$F(u) = 0.0589 + 0.5u + 0.8164u^2 - 0.4043u^4 \tag{31}$$

Proposition 4.2 gives that the solution does not blow up if $\beta T < 0.50829$ (Take $X = \infty$ with $||\psi||_\infty = 1$). Moreover, as a numerical check of (26), we have computed using a PDE solver the solution of (30) with $\psi(x) = 1$, $\tilde{F}(u) = 0.0589 + 0.5u + 0.8164u^2 + 0.4043u^4$, $\beta = 0.05$ and $T = 10$ years. The solution $X = \hat{\mathbb{P}}\left(T, -\ln \frac{|a_k|}{p_k}\right)$ coincides with our upper bound in (26) and should satisfy

$$\int_1^X \frac{ds}{-s + 0.0589 + 0.5u + 0.8164u^2 + 0.4043u^4} = 0.5 \tag{32}$$

We found $X = 4.497$ (PDE solver) and the reader can check that this value satisfies the above identity (32) as expected.

5.1. **Algorithm: Final recipe.** The algorithm for solving PDEs (5)-(6) can be described by the following steps:

(1) Choose a polynomial approximation of $u^+ \simeq \sum_{k=0}^M a_k u^k$ on the domain $[-1,1]$.
(2) Simulate the assets and the Poisson default time with intensity β (resp. $\frac{\beta}{1-R}$) for PDE2 (resp. PDE1). Note that the intensity β can be stochastic (Cox process), usually calibrated to default probabilities implied from CDS market quotes.
(3) At each default time, produce k descendants with probability p_k (given by (27)). For PDE type 2, descendants, produced after the first default, become immortal.

(4) Evaluate for each particle alive the payoff

$$\prod_{i=1}^{N_T} \psi(z_T^i) \prod_{k=0}^{M} \left(\frac{a_k}{p_k}\right)^{\omega_k}, \text{ PDE2}$$

$$\prod_{i}^{N_T\in[0,M]} \psi(z_T^i) \left(\frac{a_1(1-R)+R}{p_1}\right)^{\omega_1} \prod_{k\neq 1}^{M} \left(\frac{a_k(1-R)}{p_k}\right)^{\omega_k} \quad (\text{here, } \sum_{k=0}^{M}\omega_k = 0 \text{ or } 1), \text{ PDE1}$$

where ω_k denotes the number of branching type k. We should highlight that the algorithm for PDE1 is always convergent for all T whatever condition on the payoff as the multiplicative functional involves at most M particles.

Remark 5.1. In the case of collateralized positions, the non-linearity u_t^+ should be substituted with $(u_t - u_{t+\Delta})^+$ where Δ is a delay. Using our polynomial approximation, we get $F(u_t - u_{t+\Delta})$. By expanding this function, we get monomials of the form $\{u_t^p u_{t+\Delta}^q\}$. Our algorithm can then be easily extended to handle this case. At each default time τ, we produce p descendants starting at (τ, X_τ) and q descendants starting at $(\tau+\Delta, X_{\tau+\Delta})$.

A natural question is to characterize the error of the algorithm as a function of the approximation error of u^+ by $F(u)$. Using the parabolicity of the semi-linear PDE, we can characterize the bias of our algorithm (the proof is reported in the appendix):

Proposition 5.2. *Let us assume that* $\underline{F}(v)$ *and* $\overline{F}(v)$ *are two polynomials satisfying (***Comp***), the sufficient condition in Prop. 4.2 for a maturity* T *and*

$$\underline{F}(x) \leq x^+ \leq \overline{F}(x)$$

We denote $\underline{v}$ *and* $\overline{v}$ *the corresponding solutions of (30) and* v *the solution of (29). Then*

$$\underline{v} \leq v \leq \overline{v}$$

A similar result can be found for PDE (6). In the case of American options, our algorithm gives robust lower and upper bounds.

5.2. **Complexity.** By approximating u^+ with an infinite high-order polynomial - say N_2 - our algorithm converges towards the brute force "Monte-Carlo of Monte-Carlo" method with a complexity $O(N_1 \times N_2)$. By comparison, with our choice (31), the complexity is at most $O(4N_1)$ for PDE type (6). Moreover, this method allows to solve exactly PDE type (5), which can not be tackled without relying on an approximation within the "Monte-Carlo of Monte-Carlo" method.

5.3. **Numerical examples.** We have implemented our algorithm for the two PDE types

$$\partial_t u + \frac{1}{2}x^2\sigma_{\mathrm{BS}}^2\partial_x^2 u + \beta\left(u^+ - u\right) = 0, \quad u(T,x) = 1 - 2.1_{x>1} \ : \ \text{PDE1}$$

and

$$\partial_t u + \frac{1}{2}x^2\sigma_{\mathrm{BS}}^2\partial_x^2 u + \frac{\beta}{1-R}\left((1-R)\mathbb{E}_{t,x}[1-2.1_{X_T>1}]^+ + R\mathbb{E}_{t,x}[1-2.1_{X_T>1}] - u\right) = 0, \quad \text{PDE2}$$

with Poisson intensities $\beta = 1\%$, $\beta = 3\%$ and a recovery rate $R = 0.4$ (see Tab. 4, 5, 6, 7). In financial term, this corresponds to CDS spreads around 100 and 300 basis points. The method has been checked using a PDE solver with the polynomial approximation (31) (see Column "PDE with poly."). In order to justify the validity of (31), we have included the PDE price with the true non-linearity u^+ (see Column "PDE"). As it can be observed, prices, produced by our algorithm,

Maturity(Year)	PDE with poly.	BBM alg.	PDE
2	11.62	11.63(0.00)	11.62
4	16.54	16.53(0.00)	16.55
6	20.28	20.27(0.00)	20.30
8	23.39	23.38(0.00)	23.41
10	26.11	26.09(0.00)	26.14

Table 4. MC price quoted in percent as a function of the maturity for PDE 1 with $\beta = 1\%$.

Maturity(Year)	PDE with poly.	BBM alg.(Stdev)	PDE
2	11.62	11.64(0.00)	11.63
4	16.56	16.55(0.02)	16.57
6	20.32	20.30(0.00)	20.34
8	23.45	23.45(0.00)	23.48
10	26.20	26.18(0.00)	26.24

Table 5. MC price quoted in percent as a function of the maturity for PDE 2 with $\beta = 1\%$.

Maturity(Year)	PDE with poly.	BBM alg.	PDE
2	12.34	12.35(0.00)	12.35
4	17.72	17.71(0.00)	17.75
6	21.77	21.76(0.00)	21.82
8	25.07	25.06(0.00)	25.14
10	27.89	27.88(0.00)	27.98

Table 6. MC price quoted in percent as a function of the maturity for PDE 1 with $\beta = 3\%$.

Maturity(Year)	PDE with poly.	BBM alg.(Stdev)	PDE
2	12.38	12.39(0.00)	12.39
4	17.88	17.86(0.00)	17.91
6	22.08	22.07(0.01)	22.14
8	25.58	25.57(0.01)	25.66
10	28.62	28.60(0.01)	28.74

Table 7. MC price quoted in percent as a function of the maturity for PDE 2 with $\beta = 3\%$.

converge to the PDE solver with the polynomial approximation and are close to the exact CVA values. We would like to highlight that replacing the Black-Scholes generator $\frac{1}{2}x^2\sigma_{\mathrm{BS}}^2\partial_x^2$ by a multi-dimensional operator $\mathcal{L}$ can be easily handled in our framework by simulating the branching particles with a diffusion process associated to $\mathcal{L}$. This is out-of-reach with finite-difference scheme methods and not such an easy step for the BSDE approach.

6. Conclusion

Credit valuation adjustment is now an important quantitative issue which needs to receive special attention. The brute force "Monte-Carlo of Monte-Carlo" or the BSDE approach is not, as it looks like, a decent solution for multi-asset portfolios. We have shown the efficiency of our algorithm based on marked branching diffusions on various numerical examples. This method can also be used for semi-linear PDEs with polynomial non-linearities and extended to fully non-linear PDEs by including in the branching process Malliavin weights for derivatives. We left this investigation for future research.

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Appendix

Proof of Theorem 4.1. The proof proceeds similarly as in subsection 3.4. By using the independence and the strong Markov property, we obtain

$$\begin{aligned}
\hat{u}(t,x) &= \mathbb{E}_{t,x}[1_{\tau\geq T}\psi(z_T^1)] + \sum_{k=0}^{M}\mathbb{E}_{t,x}[1_{\tau<T}a_k\prod_{j=1}^{k}\mathbb{E}_\tau[\prod_{i=1}^{N_T^j(\tau)}\prod_{k=0}^{M}\left(\frac{a_k}{p_k}\right)^{\omega_k^j}\psi(z_T^{i,j,z_\tau})] \\
&= \mathbb{E}_{t,x}[1_{\tau\geq T}\psi(z_T^1)] + \mathbb{E}_{t,x}[1_{\tau<T}\sum_{k=0}^{M}a_k\prod_{j=1}^{k}\hat{u}(\tau,z_\tau^1)] \\
&= \mathbb{E}_{t,x}[1_{\tau\geq T}\psi(z_T^1)] + \mathbb{E}_{t,x}[F\left(\hat{u}(\tau,z_\tau^1)\right)1_{\tau<T}] \\
&= \mathbb{E}_{t,x}[e^{-\int_t^T\beta(s)ds}\psi(z_T^1)] + \int_t^T\mathbb{E}_{t,x}[\beta(s)e^{-\int_t^s\beta(u)du}F\left(\hat{u}(s,z_s^1)\right)]ds
\end{aligned}$$

By assuming that $\hat{u}\in \mathrm{L}^\infty([0,T]\times\mathbb{R}^d)$, we deduce that $\hat{u}$ is a viscosity solution of PDE (18) (see Theorem 6.4 in [17]). The comparison result (Assumption (**Comp**)) implies uniqueness, i.e. $u=\hat{u}$. □

Proof of formula 24. We set $\mathbb{P}(T|\omega)=e^{-\beta T N(\omega)}q(T|\omega)$ for convenience. We get the relation

$$q(T|\omega)=\beta\sum_{k=0}^{M}\int_0^T dtq(t|\omega_0,\dots,\omega_k-1,\dots,\omega_M)N(\omega_0,\dots,\omega_k-1,\dots,\omega_M)p_k e^{\beta t(k-1)}$$

which is equivalent to

$$\begin{aligned}
\partial_T q(T|\omega) &= \beta\sum_{k=0}^{M}q(T|\omega_0,\dots,\omega_k-1,\dots,\omega_M)N(\omega_0,\dots,\omega_k-1,\dots,\omega_M)p_k e^{\beta T(k-1)} \\
q(0|\omega) &= \delta_{\omega=0}
\end{aligned}$$

The Laplace transform of q, $\hat{q}(T,c)\equiv\mathbb{E}^q[\prod_{k=0}^{M}e^{-(k-1)c_k\omega_k}]$, satisfies the first-order PDE

$$\partial_T\hat{q}(T|c)=\beta\sum_{k=0}^{M}p_k\left(\hat{q}(T,c)-\sum_{q=0}^{M}\partial_{c_q}\hat{q}(T,c)\right)e^{(\beta T-c_k)(k-1)}\ ,\ \hat{q}(0|c)=1$$

The solution is given by

$$\hat{q}(T|c)=e^{(c_0-c_0(T))}$$

where the coefficients $\{c_q(T)\}_{q=0,\ldots,M}$ are solutions of the ODEs

$$\frac{dc_q(t)}{dt} = -\beta \sum_{k=0}^{M} p_k e^{(\beta(T-t)-c_k(t))(k-1)} \ , \ c_q(0) = c_q$$

The solution is given by $c_q(t) = c_q - \beta t - \ln U(t|c)$ with

$$\frac{dU(t|c)}{dt} = \beta \left(-U(t|c) + \sum_{k=0}^{M} p_k e^{(\beta T - c_k)(k-1)} U^k(t|c) \right) \ , \ U(0|c) = 1$$

This gives

$$\begin{aligned} \hat{q}(T|c) &= e^{\beta T} U(T|c) \ \text{ if } \sum_{k=0}^{M} p_k e^{(\beta T - c_k)(k-1)} \neq 1 \\ &= e^{\beta T} \ \text{ if } \sum_{k=0}^{M} p_k e^{(\beta T - c_k)(k-1)} = 1 \end{aligned}$$

where $U(T|c)$ satisfies

$$\int_1^{U(T|c)} \frac{ds}{-s + \sum_{k=0}^{M} p_k e^{(\beta T - c_k)(k-1)} s^k} = \beta T \text{ if } \sum_{k=0}^{M} p_k e^{(\beta T - c_k)(k-1)} \neq 1$$

Finally, we use that $\hat{\mathbb{P}}(T|c) = U(T|\frac{c_k}{k-1} + \beta T)$. □

Proof of Proposition 5.2. The function $\delta = \bar{v} - v$ satisfies the linear PDE

$$\partial_t \delta + \mathcal{L}\delta - \beta\delta + \beta \left(\frac{\bar{v}^+ - v^+}{\bar{v} - v} \right) 1_{v \neq \bar{v}} \delta + \beta \left(F(\bar{v}) - \bar{v}^+ \right) = 0 \ , \ \delta(T, x) = 0$$

Note that the term $r_t \equiv 1 - \left(\frac{\bar{v}_t^+ - v_t^+}{\bar{v}_t - v_t} \right) 1_{v_t \neq \bar{v}_t}$ is lower bounded. Feynman-Kac's formula gives

$$\delta(t, x) = \int_t^T \beta \mathbb{E}_{t,x}[\left(F(\bar{v}) - \bar{v}^+ \right) e^{-\beta \int_t^s r_u du}]$$

from which we conclude the proof as $\bar{F}(x) \geq x^+$ by assumption. □

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Société Générale, Global Market Quantitative Research

E-mail address: pierre.henry-labordere@sgcib.com