ON THE DIVERGENCE OF BIRKHOFF NORMAL FORMS

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To the memory of my father Grégoire Krikorian (1934–2018)

ABSTRACT

It is well known that a real analytic symplectic diffeomorphism of the $2d$-dimensional disk ($d \geq 1$) admitting the origin as a non-resonant elliptic fixed point can be *formally* conjugated to its Birkhoff Normal Form, a formal power series defining a *formal integrable* symplectic diffeomorphism at the origin. We prove in this paper that this Birkhoff Normal Form is in general divergent. This solves, in any dimension, the question of determining which of the two alternatives of Pérez-Marco's theorem (Ann. Math. (2) 157:557–574, 2003) is true and answers a question by H. Eliasson. Our result is a consequence of the fact that when $d = 1$ the convergence of the formal object that is the BNF has strong dynamical consequences on the Lebesgue measure of the set of invariant circles in arbitrarily small neighborhoods of the origin. Our proof, as well as our results, extend to the case of real analytic diffeomorphisms of the annulus admitting a Diophantine invariant torus.

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1. Introduction

We consider in this paper *real analytic* diffeomorphisms defined on an open set of the $2d$-cartesian space $\mathbf{R}^d\times\mathbf{R}^d$ or respectively of the $2d$-cylinder (or annulus) $(\mathbf{R}/2\pi\mathbf{Z})^d\times\mathbf{R}^d$ ($d\geq 1$), which are *symplectic* with respect to the canonical symplectic forms $\sum_{j=1}^d dx_j\wedge dy_j$, $(x,y)\in\mathbf{R}^d\times\mathbf{R}^d$, resp. $\sum_{j=1}^d d\theta_j\wedge dr_j$, $(\theta,r)\in(\mathbf{R}/2\pi\mathbf{Z})^d\times\mathbf{R}^d$, and leave invariant $\{(0,0)\}\in\mathbf{R}^d\times\mathbf{R}^d$, resp. the torus $\mathcal{T}_0:=(\mathbf{R}/2\pi\mathbf{Z})^d\times\{0\}\subset(\mathbf{R}/2\pi\mathbf{Z})^d\times\mathbf{R}^d$. We shall assume that the invariant sets $\{(0,0)\}\in\mathbf{R}^d\times\mathbf{R}^d$, resp. $(\mathbf{R}/2\pi\mathbf{Z})^d\times\{0\}$, are *elliptic equilibrium sets* in the following sense: there exists $\omega=(\omega_1,\ldots\omega_d)\in\mathbf{R}^d$, the *frequency vector*, such that

$$(\mathbf{1.1})\qquad \begin{cases} f:(\mathbf{R}^d\times\mathbf{R}^d,(0,0))\circlearrowleft, & f=Df(0,0)\circ(id+O^2(x,y))\\ \mathrm{spec}(Df(0,0))=\{e^{\pm 2\pi\sqrt{-1}\omega_j},\ 1\leq j\leq d\}\end{cases}$$

and respectively

$$(\mathbf{1.2})\qquad f:((\mathbf{R}/2\pi\mathbf{Z})^d\times\mathbf{R}^d,\mathcal{T}_0)\circlearrowleft,\qquad f(\theta,r)=(\theta+2\pi\omega,r)+(O(r),O(r^2)).$$

If $\omega_i\neq\omega_j$ for $i\neq j$ (stronger non-resonance condition will be made later), the derivative $Df(0,0)$ of f at the fixed point $(0,0)$ in (1.1) can be symplectically conjugated to a *symplectic rotation* and we can thus assume $Df(0,0)$ is a symplectic rotation: for any $x=(x_1,\ldots,x_d), y=(y_1,\ldots,y_d), \widetilde{x}=(\widetilde{x}_1,\ldots,\widetilde{x}_d),\widetilde{y}=(\widetilde{y}_1,\ldots,\widetilde{y}_d)$ one has ($i=\sqrt{-1}$)

$$Df(0,0)\cdot(x,y)=(\widetilde{x},\widetilde{y})\iff\begin{cases}\widetilde{x}_j+i\widetilde{y}_j=e^{2\pi i\omega_j}(x_j+iy_j)\\ \forall\ 1\leq j\leq d.\end{cases}$$

We shall refer to situation (1.1) as the *Elliptic fixed point* or the *Cartesian Coordinates* ((CC) for short) case and to situation (1.2) as the *Action-Angle* ((AA) for short) case.

Important examples of such diffeomorphisms are provided by flows $(\Phi_H^t)_{t\in\mathbf{R}}$, or by suitable Poincaré sections on some energy level, of Hamiltonian systems

$$\dot{x}=\frac{\partial H}{\partial y}(x,y),\quad \dot{y}=-\frac{\partial H}{\partial x}(x,y),\quad \text{resp.}\quad \dot{\theta}=\frac{\partial H}{\partial r}(\theta,r),\quad \dot{r}=-\frac{\partial H}{\partial\theta}(\theta,r)$$

where $H:(\mathbf{R}^{d'}\times\mathbf{R}^{d'},(0,0))\to\mathbf{R}$ resp. $H:((\mathbf{R}/2\pi\mathbf{Z})^{d'}\times\mathbf{R}^{d'},\mathcal{T}_0)\to\mathbf{R}$ ($d'=d$ or $d'=d+1$) is real analytic and satisfies

$$(\mathbf{1.3})\qquad \text{(CC)-case}\qquad H(x,y)=2\pi\sum_{j=1}^{d'}\omega_j\frac{x_j^2+y_j^2}{2}+O^3(x,y),$$

$$(\mathbf{1.4})\qquad \text{(AA)-case}\qquad H(\theta,r)=2\pi\sum_{j=1}^{d'}\omega_j r_j+O(r^2).$$

If we denote by Φ_H the time-1 map of a Hamiltonian H and define the observable $r_j : (x, y) \mapsto (1/2)(x_j^2 + y_j^2)$, resp. $r_j : r \mapsto r_j$ $(1 \le j \le d)$ we can write (1.1), resp. (1.2), as

$$\textbf{(1.5)} \qquad \text{(CC)-case} \quad f : (\mathbf{R}^d \times \mathbf{R}^d, (0,0)) \circlearrowleft, \qquad f = \Phi_{2\pi\langle\omega,r\rangle} + O^2(x,y)$$

$$\textbf{(1.6)} \qquad \text{(AA)-case} \quad f : ((\mathbf{R}/2\pi\mathbf{Z})^d \times \mathbf{R}^d, \mathcal{T}_0) \circlearrowleft, \qquad f = \Phi_{2\pi\langle\omega,r\rangle} + (O(r), O(r^2))$$

where $\langle\omega, r\rangle = \sum_{j=1}^d \omega_j r_j$, $r = (r_1, \ldots, r_d)$.

The representations (1.5), resp. (1.6), give a very rough understanding of the behavior of the *finite time* dynamics of the diffeomorphism f in a neighborhood of the elliptic equilibrium sets $\{(0, 0)\}$, resp. $\mathcal{T}_0$: it is interpolated[1] by the dynamics of $\Phi_{2\pi\langle\omega,r\rangle}$ which is *quasi-periodic* in the sense that all its orbits are quasi-periodic with frequencies $\omega_1, \ldots, \omega_d$. Improving this approximation is an old and important problem (it was a central theme of research of the astronomers of the XIXth century; see the references of the very instructive introduction by Pérez-Marco in [36]) that has a solution at least in the (CC)-case (1.5) if the frequency vector ω is *non-resonant*: any relation $k_0 + k_1\omega_1 + \cdots + k_d\omega_d = 0$ with $k_0, k_1, \ldots, k_d \in \mathbf{Z}$ implies that $k_0 = k_1 = \cdots = k_d = 0$. Indeed, after using nice changes of coordinates (symplectic transformations) one can interpolate, in *small neighborhoods of the origin*, the dynamics of f by quasi-periodic ones with much better orders of approximation and for much longer times. There are two remarkable features of this interpolation: the first, is that the frequencies of the interpolating quasi-periodic motions now depend on the initial point and do not necessarily coincide with the frequencies at the origin; the second, is that if we push the order of approximation, these frequencies stabilize in a way. This is the content of the famous *Birkhoff Normal Form Theorem*, formalized by Birkhoff in the 1920's [5], [4], [49] and which paved the way to the major achievements of the KAM theory (named after Kolmogorov, Arnold and Moser) in the 1960's, on the existence of (infinite time) quasi-periodic motions for a wide class of diffeomorphisms of the form (1.1), (1.2); see [29], [1], [32] (and [34] for finite time approximations). We now describe in more details the Birkhoff Normal Form Theorem.

1.1. *Birkhoff Normal Forms.* — From now on, we assume that ω is non-resonant.

We begin with the Elliptic fixed point case ((CC)-case). The first statement of the Birkhoff Normal Form Theorem is the following. For any $N \in \mathbf{N}^*$, there exist a polynomial $B_N \in \mathbf{R}[r_1, \ldots, r_d]$, $B_N(r) = 2\pi\langle\omega, r\rangle + O(r^2)$, of total degree N and a symplectic diffeomorphism $Z_N : (\mathbf{R}^d \times \mathbf{R}^d, (0, 0)) \circlearrowleft$ (preserving the standard symplectic form $\sum_{k=1}^d dx_k \wedge dy_k$ and tangent to the identity $Z_N = id + O^2(x, y)$) such that

$$\textbf{(1.7)} \qquad Z_N \circ f \circ Z_N^{-1}(x,y) = \Phi_{B_N}(x,y) + \mathcal{O}^{2N+1}(x,y).$$

[1] For (x, y) ε-close to $(0, 0)$ and $n \in \mathbf{N}$ not too large $n = O(\varepsilon^{-\alpha})$, $0 < \alpha < 1$ the iterates $f^k(x, y)$, $k \le n$ (f^k denotes the composition $f \circ \cdots \circ f$, k times) stay $\varepsilon^{2-\alpha}$-close to those of the symplectic rotation, $\Phi^k_{2\pi\langle\omega,r\rangle}(x, y)$.

The diffeomorphism $\Phi_{B_N} : (\mathbf{R}^d \times \mathbf{R}^d, 0) \circlearrowleft$ is a *generalized symplectic rotation*

$$\Phi_{B_N}(x,y) = (\widetilde{x},\widetilde{y}) \iff \begin{cases} \widetilde{x}_j + i\widetilde{y}_j = e^{i\partial_j B_N(r)}(x_j + iy_j) \\ \forall\ 1 \leq j \leq d \end{cases} \tag{1.8}$$

(recall $r = ((1/2)(x_1^2 + y_1^2), \ldots, (1/2)(x_d^2 + y_d^2)))$ and defines an *integrable* dynamics in a strong sense: every orbit of Φ_{B_N} is *quasi-periodic* and, in addition, the origin is *Lyapunov stable*. Indeed, for each $c = (c_1, \ldots, c_d) \in (\mathbf{R}_+^*)^d$, the d-dimensional torus

$$\mathcal{T}_c := \{(x,y) \in \mathbf{R}^{2d},\ \forall\ 1 \leq j \leq d,\ r_j := (1/2)(x_j^2 + y_j^2) = c_j\}$$

is globally invariant by Φ_{B_N} and the restricted dynamics of Φ_{B_N} on the torus $\mathcal{T}_c \simeq \mathbf{T}^d := \mathbf{R}^d/(2\pi\mathbf{Z})^d$ is conjugated to a translation $\mathbf{T}^d \ni \theta \mapsto \theta + 2\pi\omega(c) \in \mathbf{T}^d$ with *frequency vector* $\omega(c) = (2\pi)^{-1}\nabla B_N(c)$. The dynamics of Φ_{B_N} is thus completely understood on the whole phase space[2] $\mathbf{R}^d \times \mathbf{R}^d$.

Here comes the second part of the statement. The polynomials B_N and the components of $Z_N - id$ converge as *formal* power series when N goes to infinity: $B_N \to B_\infty \in \mathbf{R}[[r_1, \ldots, r_d]]$, $Z_N \to Z_\infty \in \mathbf{R}[[x,y]]$ and, in the set of formal power series $\mathbf{R}[[x,y]]$, one has the following formal conjugacy relation

$$Z_\infty \circ f \circ Z_\infty^{-1}(x,y) = \Phi_{B_\infty}(x,y). \tag{1.9}$$

The formal power series B_∞ is unique if Z_∞ is tangent to the identity and is therefore *invariant by (smooth or formal) conjugations* tangent to the identity; it is called the *Birkhoff Normal Form* (BNF for short) of f and we shall denote it by BNF(f):

$$\mathrm{BNF}(f) = B_\infty(r_1, \ldots, r_d) \in \mathbf{R}[[r_1, \ldots, r_d]].$$

On the other hand the formal conjugacy Z_∞, which is called the *normalization transformation*, is not unique (but if properly normalized is unique).

The preceding results hold in the Action-Angle case (1.6) but under a *Diophantine assumption* on ω (this is stronger than mere non-resonance):

$$\forall\ k \in \mathbf{Z}^d \setminus \{0\},\ \min_{l \in \mathbf{Z}} |\langle k, \omega\rangle - l| \geq \frac{\kappa}{|k|^\tau} \qquad (\tau \geq d). \tag{1.10}$$

The positive numbers τ and κ are called respectively the *exponent* and the *constant* of the Diophantine condition.[3] One can then prove similarly the existence: (a) for any $N \in \mathbf{N}^*$,

[2] When c has some zero components, $\mathcal{T}_c$ is a d_c-dimensional torus, $0 \leq d_c \leq d$, and the restricted dynamics of Φ_{B_N} on $\mathcal{T}_c$ is again conjugate to a translation on a torus.

[3] The set of vectors of $\mathbf{R}^d$ satisfying a Diophantine condition with fixed exponent τ and fixed constant κ has positive Lebesgue measure if $\tau > d$ and if $\kappa > 0$ is small enough; for each $\tau > d$, the union of these sets on all $\kappa > 0$ has full Lebesgue measure in $\mathbf{R}^d$.

of a polynomial $B_N \in \mathbf{R}[r_1, \dots, r_d]$, $B_N(r) = 2\pi\langle\omega, r\rangle + O(r^2)$ and of a symplectic diffeomorphism $Z_N : ((\mathbf{R}/2\pi\mathbf{Z})^d \times \mathbf{R}^d, \mathcal{T}_0) \circlearrowleft$ (preserving the standard symplectic form $\sum_{k=1}^d d\theta_k \wedge dr_k$, $Z_N = id + (O(r), O(r^2))$) such that

$$Z_N \circ f \circ Z_N^{-1}(\theta, r) = \Phi_{B_N}(\theta, r) + (\mathcal{O}^N(r), \mathcal{O}^{N+1}(r)) \tag{1.11}$$

$$\Phi_{B_N}(\theta, r) = (\theta + \nabla B_N(r), r) \tag{1.12}$$

(Φ_{B_N} is called an *integrable twist*); and: (b) of a formal power series $B_\infty \in \mathbf{R}[[r_1, \dots, r_d]]$, the Birkhoff Normal Form, and of a formal symplectic transformation $Z_\infty = id + (O(r), O(r^2))$ in $C^\omega(\mathbf{T}^d)[[r_1, \dots, r_d]]$ (the set of formal power series with coefficients in the set of real analytic functions $\mathbf{T}^d \to \mathbf{T}$) such that one has in $C^\omega(\mathbf{T}^d)[[r_1, \dots, r_d]]$ the formal conjugation relation

$$Z_\infty \circ f \circ Z_\infty^{-1}(\theta, r) = (\theta + \nabla B_\infty(r), r). \tag{1.13}$$

Again we denote $\mathrm{BNF}(f) = B_\infty(r_1, \dots, r_d) \in \mathbf{R}[[r_1, \dots, r_d]]$.

All the preceding discussion on Birkhoff Normal Forms holds if we only assume f to be smooth.[4] We can summarize this:

Theorem (Birkhoff). — Any smooth symplectic diffeomorphism $f : (\mathbf{R}^d \times \mathbf{R}^d, (0,0)) \circlearrowleft$ ($d \geq 1$) (resp. $f : ((\mathbf{R}/2\pi\mathbf{Z})^d \times \mathbf{R}^d, \mathcal{T}_0) \circlearrowleft$) admitting the origin as a non-resonant elliptic fixed point (resp. of the form (1.6) with ω Diophantine) is formally (strongly) integrable: it is conjugated in $\mathbf{R}[[x, y]]$ (resp. $C^\infty((\mathbf{R}/2\pi\mathbf{Z})^d)[[r]]$) to the formal generalized symplectic rotation (resp. the formal integrable twist) $\Phi_{\mathrm{BNF}(f)}$. The formal series $\mathrm{BNF}(f)$ is an invariant of formal conjugation.

We refer to [4] and [49] (Section 24) for a proof of the preceding theorem in the case of symplectic diffeomorphisms of the disk admitting a non-resonant elliptic fixed point and to [13] for the case of Hamiltonian systems admitting a Diophantine KAM torus. We shall reformulate (in the real analytic case) this theorem in Section 6, *cf.* Propositions 6.1–6.2, and shall give a proof of it in Section E of the Appendix where we mainly concentrate on the AA-case.

These formal (and approximate) Birkhoff Normal Forms can be defined in the more classical setting of Hamiltonian flows $\dot{x} = \frac{\partial H}{\partial y}(x, y)$, $\dot{y} = -\frac{\partial H}{\partial x}(x, y)$ (or $\dot{\theta} = \frac{\partial H}{\partial r}(\theta, r)$, $\dot{r} = -\frac{\partial H}{\partial \theta}(\theta, r)$): f and Φ_B ($B = \mathrm{BNF}(f)$) are then replaced by $(\Phi_H^t)_{t \in \mathbf{R}}$ and $(\Phi_B^t)_{t \in \mathbf{R}}$ in (1.9), (1.13) (we shall then write $B = \mathrm{BNF}(H)$[5]).

In the Hamiltonian case, there is a weaker notion of integrability, usually called *Poisson integrability*, which corresponds to the situation where the considered Hamiltonian has a complete system of functionally independent integrals (observables constant under the motion) which commute for the Poisson bracket. Poincaré discovered [37] that, in

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[4] In the C^k category, one can define B_N and Z_N up to some order N depending on k but one cannot define in general $\mathrm{BNF}(f)$.

[5] A more classic equivalent formulation is $H = B \circ Z$.

general, real analytic Hamiltonian flows do not admit other analytic first integrals than the Hamiltonian itself and hence that in general no relation like (1.9) can hold with converging Z_∞ and B_∞. Siegel proved [48] in 1954 (see also [47], [49], [52], [36]) that, whatever the *fixed* non-resonant frequency vector at the origin ω is, the normalizing conjugation Z_∞ cannot in general[6] define a convergent series. Indeed, the existence of a convergent normalizing transformation yields real analytic Poisson integrability[7] a fact (known to Birkhoff [5]) that is not compatible with the richness[8] of a generic dynamics near a non-resonant elliptic equilibrium. Note that the converse statement is true: real analytic Poisson integrability implies the existence of a real analytic normalizing Birkhoff transformation (*cf.* [25], [28], [56]).

As for the Birkhoff Normal Form itself, H. Eliasson formulated the following natural question [11], [10] (see also the references in [36]):

Question **A** *(Eliasson). — Are there examples of real analytic symplectic diffeomorphisms or Hamiltonians admitting divergent (i.e. with a null radius of convergence) Birkhoff Normal Form?*

The preceding question has an easy positive answer in the smooth case (the map f is only assumed to be smooth): indeed, one can choose f to be of the form $f = \Phi_\Omega$ where $\Omega : (\mathbf{R}^d, 0) \to \mathbf{R}$ is smooth with a divergent Taylor series at the origin; since equalities (1.9) (1.13) only depend on the infinite jet $J(f)$ of f at 0, the special integrable form of f implies $BNF(f) = J(f)$ thus $BNF(f)$ is diverging. The situation is not so clear if f is real analytic. In contrast with the aforementioned generic divergence of the normalizing transformation, there seems[9] to be *a priori* no obvious dynamical obstruction[10] to the divergence of the Birkhoff Normal Form.

The first breakthrough in connection with Eliasson's question came from R. Pérez-Marco [36] who proved, in the setting of Hamiltonian systems having a non-resonant elliptic fixed point, the following *dichotomy*:

Theorem (Pérez-Marco [36]). — For any fixed non-resonant *frequency vector* $\omega \in \mathbf{R}^{d'}$, $d' \geq 2$, *one has the following dichotomy: either for all real analytic Hamiltonian* H *of the form (1.3)* $BNF(H)$ *converges (defines a converging analytic series) or there is a "prevalent" set of such* H *for which* $BNF(H)$ *diverges.*

We refer to Section 1.4 for a precise definition of "prevalent". A similar dichotomy holds in the setting of real analytic symplectic diffeomorphisms in the (CC)-case, and,

[6] Here it means G_δ-dense in some set of real analytic functions with fixed radius of convergence. This phenomenon is even "prevalent" as shown by Pérez-Marco [36].

[7] If Z_∞ converges the observables $r_j \circ Z_\infty, j = 1, \ldots, d'$ are a complete set of real analytic and functionally independent Poisson commuting integrals.

[8] By which we mean the coexistence of quasi-periodic motions and hyperbolic behavior in any neighborhood of the equilibrium; see for a global view on these topics and references the book [2].

[9] We shall in fact see in this paper that *there are* such dynamical obstructions.

[10] Like the accumulation at the origin of hyperbolic periodic points or normally hyperbolic tori.

both in the Hamiltonian or diffeomorphism framework, it can be extended to the (AA)-case (but under the stronger assumption that ω is Diophantine); *cf.* Theorem 1.3 of our paper.

Pérez-Marco's argument is not based on an analysis of the *dynamics* of f but rather focuses on the *coefficients* of the BNF and exploits their polynomial dependence on the coefficients of the initial perturbation by using techniques from potential theory.[11]

The following two Theorems are an answer (in the symplectomorphism setting) to Eliasson's question and decide which of the two assertions of Pérez-Marco's alternative holds (see Theorem D of Section 1.4 for a more precise statement).

Main Theorem **1** *((CC)-Case). — For any $d \geq 1$ and any non-resonant frequency vector $\omega \in \mathbf{R}^d$, there exists a "prevalent" set of real analytic symplectic diffeomorphism $f : (\mathbf{R}^d \times \mathbf{R}^d, (0,0)) \circlearrowleft$ of the form (1.5) the Birkhoff Normal Forms of which are divergent.*

In the Action-Angle Case (1.6) it takes the following form:

Main Theorem **1'** *((AA)-Case). — For any $d \geq 1$ and any Diophantine frequency vector $\omega \in \mathbf{R}^d$, there exists a "prevalent" set of real analytic symplectic diffeomorphism $f : ((\mathbf{R}/2\pi\mathbf{Z})^d \times \mathbf{R}^d, \mathcal{T}_0)$ of the form (1.6) the Birkhoff Normal Forms of which are divergent.*

Main Theorems 1, 1' also extend to the Hamiltonian case (1.3)–(1.4) (with $d' = d+1$).[12] Note that from Pérez-Marco's Theorem (and its analogue in the symplectomorphism case), in order to prove that the divergence of the Birkhoff Normal Form holds in a prevalent way, it is enough to provide, for each fixed frequency vector ω, one example for which the BNF is divergent. On the other hand, if one is able to construct one such example for some d_0, then it is easy to construct other such examples for any $d > d_0$ (see for example the proof of Theorem D). Proving Main Theorem 1 (resp. 1') thus amounts to constructing when $d = 1$, for each irrational (resp. Diophantine) $\omega \in \mathbf{R}$, one example of a real analytic symplectic diffeomorphism with a diverging BNF.

Gong already provided in [17] (by a direct analysis of the coefficients of the BNF) examples of real analytic Hamiltonians $\langle \omega, r \rangle + \mathrm{F} : (\mathbf{R}^2 \times \mathbf{R}^2, 0) \to \mathbf{R}$, $\mathrm{F} = \mathrm{O}^3(x, y)$, with Liouvillian frequency $\omega \in \mathbf{R}^2$ at the origin and a divergent BNF and Yin [54] produced analogue of Gong's examples in the diffeomorphism case (area preserving map of $(\mathbf{R}^2, 0)$ with a very Liouvillian elliptic fixed point). In these examples the divergence of the BNF is caused by the presence of very small denominators (due to the Liouvillian character of ω) appearing in the coefficients of the BNF. After our result was announced, Fayad [15] constructed simple examples of real analytic Hamiltonian systems in $(\mathbf{R}^8, 0)$ ($d' = d + 1 = 4$

[11] The idea of using potential theory in problems of small denominators was first introduced by Yu. Ilyashenko [23]. See [35] for further references.

[12] It is not clear whether one can, for general systems, deduce the case of Hamiltonian flows from the case of diffeomorphisms and *vice versa*. On the other hand the proofs of Main Theorems 1, 1' and in particular the proofs of Main Theorem 2 and of Theorems A–B, A'–B' below extend to Hamiltonian flows with $1+1$ degrees of freedom.

degrees of freedom) with any fixed non-resonant frequency vector at the origin and divergent BNF. The argument again is based on an analysis of the coefficients of the BNF; one considers Hamiltonians with two degrees of freedom where two extra action variables are added as formal parameters, one of them appearing later in the denominators of the BNF. These types of examples can be constructed in the diffeomorphism case for $d \geq 3$. In a different context, that of reversible systems, let us mention a result of divergence of normal forms in [19] based on a different method (control of coefficients growth) and a result of divergence of normalizing transformations in [33].

We now formulate Eliasson's question in a stronger form:

Question **B.** — *Does the convergence of a* formal *conjugacy invariant like the Birkhoff Normal Form of a real analytic symplectic diffeomorphism (or Hamiltonian) have consequences on the* dynamics *of the diffeomorphism (or Hamiltonian)?*

Note that the convergence of the normalizing transformation has an obvious consequence, namely, integrability. As for Question B, there are various results pointing to some kind of rigidity phenomena if analyticity (and some arithmetic properties on ω) is assumed. To be more specific, let us mention a striking one: Bruno [7] and Rüssmann [42] proved that if f is real analytic and if its BNF is *trivial*, $\mathrm{BNF}(f) = 2\pi\langle\omega, r\rangle$ (in particular $\mathrm{BNF}(f)$ converges), then f is real analytically conjugated to $\Phi_{2\pi\langle\omega,r\rangle}$, *provided* the frequency vector at the origin ω satisfies a *Diophantine condition*. We refer to [53], [25], [11], [9], [51], [18], [13], [12] for generalizations of the Bruno-Rüssmann Theorem and related results.

The Main Result of our paper is in some sense one answer, amongst possibly others, to the previous question at least when $d = 1$ and if f is assumed to satisfy some *twist condition*.

Let us say that a diffeomorphism of the form $(\mathbf{R}^2, 0) \circlearrowleft$ (1.1) or $(\mathbf{R} \times \mathbf{T}, \mathcal{T}_0) \circlearrowleft$ is *twist* (or satisfies a *twist condition*) if the second order term of its BNF is not zero:[13] $(2\pi)^{-1}\mathrm{BNF}(f)(r) = \omega r + b_2 r^2 + O(r^3)$, $b_2 \neq 0$.

Main Theorem **2.** — *If the Birkhoff Normal Form of a real analytic symplectic twist diffeomorphism* $(\mathbf{R}^2, 0) \circlearrowleft$ *(1.1) or* $(\mathbf{R} \times \mathbf{T}, \mathcal{T}_0) \circlearrowleft$ *(1.2) converges then the measure of the complement of the union of all invariant curves accumulating the origin is much smaller than what it is for a general such diffeomorphism.*

In other words, the convergence of a *formal* object like the BNF has consequences on the *dynamics* of the diffeomorphism. Precise statements are given in Theorems A–B, A'–B') of Section 1.2 and Theorems E and E' of Section 1.4. Combined with (the extension to the diffeomorphism case of) Pérez-Marco's Theorem [36], this gives that in any number of degrees of freedom, a general real analytic symplectic diffeomorphism admit-

[13] An easily checkable condition.

ting the origin as an non-resonant elliptic equilibrium has a divergent Birkhoff Normal Form (see Theorem D).

Having in mind the aforementioned result by Bruno and Rüssmann, a natural stronger question is whether the following *rigidity* result is true:

Question **C.** — *Is it true that a real analytic symplectic diffeomorphism or Hamiltonian system having a Diophantine elliptic equilibrium and a non degenerate and convergent BNF is (real analytically) integrable (in some neighborhood of the origin)?*

The examples by Farré and Fayad in [14] of real analytic Hamiltonians on $\mathbf{T}^{d+1} \times \mathbf{R}^{d+1}$ with convergent BNF and with an unstable Diophantine elliptic torus show that such a generalization is not true for $d \geq 2$, at least in the (AA) case[14] and if by non degenerate we mean that BNF(f) is not trivial. The question is still open for $d < 2$. Note that though in Farré-Fayad's examples the BNF (which is explicit) is not degenerate (the rank of its quadratic part is not zero unlike in the Bruno-Rüssmann Theorem), its quadratic part is not of maximal rank. If one drops in Question C the Diophantine assumption and assumes the BNF to be trivial (like in Bruno-Rüssmann's Theorem) the question is open (this question is related to a question of Birkhoff on pseudo-rotations[15] and to the problem of constructing real analytic Anosov-Katok examples; *cf.* [16] for details and references).

When $d = 1$ the situation might be more favorable. To any twist area preserving diffeomorphism $f : (\mathbf{R}^2, 0) \circlearrowleft$ (1.1) or $f : (\mathbf{R} \times \mathbf{T}, \mathcal{T}_0) \circlearrowleft$ (1.2) one can associate (we use the notations and terminology of [46]) its *minimal action* $\alpha : \mathrm{I} \to \mathbf{R}$ (I is an open interval containing ω) that assigns to each $\varphi \in \mathrm{I}$ the average action of any minimal orbit with rotation number φ. The function α is strictly convex (in fact differentiable at any irrational) and one can thus define its Legendre conjugate function $\alpha^* : r \mapsto \sup_{\varphi \in \mathrm{I}}(\varphi r - \alpha(\varphi))$ [16] (see [30], [31], [46] for further details). The function $r \mapsto \alpha^*(r)$ (defined on a neighborhood of 0) can be seen as a *frequency map* in the sense that if γ is an invariant circle for f with "symplectic height" (area with respect to the origin) c then $\alpha^*(c)$ is the rotation number of f restricted on γ. It has the following properties: the Taylor series of α^* at 0 coincides with the Birkhoff Normal Form of f; moreover, if α^* (hence α) is differentiable then f is C^0-integrable (see [46]). This C^0-integrability often yields rigidity (we refer to [3], [27] for an illustration of this fact in the context of billiard maps). The techniques developed in our paper are probably enough to prove that if the function α^* is real analytic then f is in fact real analytically integrable. A more delicate issue is to establish real analyticity of α^* by only knowing that its Taylor series at 0 (the BNF) defines a converging series. Note that if f is real analytic one can construct dynamically relevant holomorphic functions (frequency maps) defined on complex domains having positive Lebesgue measure intersections (Cantor sets) with the real axis (see [8], [38]) and which coincide

[14] It seems that the (CC) case is not yet settled.
[15] Area preserving maps with no periodic points except the origin.
[16] The functions α and α^* (also denoted β and α) are called Mather's functions.

on these intersections with α^*. The restrictions of these holomorphic functions on these Cantor sets have some quasi-analyticity properties but it seems that there are not strong enough to deduce that α^* behaves like a genuine quasi-analytic function (in particular that the convergence of the Taylor series at 0 implies analyticity); we refer to [8] for references and for more details.

We conclude this subsection by the following question.

Question **D.** — *Is a given real analytic symplectic diffeomorphism accumulated*[17] *by real analytic symplectic diffeomorphisms having convergent BNF's? (We do not ask the radii of convergence of the BNF's to be bounded below).*

Positive answers to Questions C, D would imply that any real analytic symplectic diffeomorphism admitting an elliptic equilibrium set is accumulated in the strong real analytic topology by diffeomorphisms of the same type that are in addition integrable in a neighborhood of the equilibrium set.

1.2. *Invariant circles.* — As suggest (1.7), (1.11) the BNF (more precisely its approximate version B_N) is, as we have already mentioned, a precious tool to study the problem of the existence of quasi-periodic motions in the neighborhood of an elliptic equilibrium. A bright illustration of this fact is certainly the KAM Theorem ([29], [1], [32]) that yields, under suitable *non-degeneracy* conditions on the BNF (non-planarity), the existence of many KAM tori[18] accumulating the origin (see [13], [12] for results under much weaker non-degeneracy assumptions).

We shall be mainly concerned with the 2-dimensional case ($d = 1$) and we restrict to this case in this subsection. Recall our notation $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ for the 1-dimensional torus. An *invariant circle* (or also *invariant curve*) for a real analytic (or smooth) diffeomorphism $f : (\mathbf{R} \times \mathbf{R}, (0,0)) \circlearrowleft$ of the form (1.5) is the image $\gamma = g(\mathbf{T})$ of an injective C^1 map $g : \mathbf{T} \to \mathbf{R}^2 \setminus \{0\}$ with index ± 1 at 0 such that $f(\gamma) = \gamma$. Likewise, in the (AA) case, an *invariant circle* or *invariant curve*) for a real analytic (or smooth) diffeomorphism $f : (\mathbf{T} \times \mathbf{R}, \mathcal{T}_0) \circlearrowleft$ of the form (1.6) is the image $\gamma = g(\mathbf{T})$ of an injective C^1 map $g : \mathbf{T} \to \mathbf{T} \times \mathbf{R}$ which is homotopic to the circle $\mathcal{T}_0 = \mathbf{T} \times \{0\}$ and such that $f(\gamma) = \gamma$.[19] Note that in this latter case, by a theorem of Birkhoff (*cf.* [4], [21]), invariant circles close enough to $\mathcal{T}_0$ are in fact *graphs* if f satisfies a *twist condition*:

$$(\mathbf{1.14}) \qquad b_2(f) \neq 0 \qquad \text{if} \quad (2\pi)^{-1}\mathrm{BNF}(f) = \omega r + b_2(f) r^2 + \cdots .$$

[17] This means that if the given diffeomorphism f has a holomorphic extension to some complex domain W there exists a slightly smaller subdomain $\widetilde{W} \subset W$ and a sequence of real-symmetric holomorphic diffeomorphisms f_n defined on $\widetilde{W}$ such that $\lim_{n\to\infty} \sup_{\widetilde{W}} |f - f_n| = 0$.

[18] A KAM torus is an invariant Lagrangian torus on which the dynamics is conjugated to a linear translation with a Diophantine frequency vector.

[19] These curves are also called *essential curves*.

In both cases, we denote by $\overline{\mathcal{G}}_f$ the set of f-invariant curves and, for $t > 0$, by $\overline{\mathcal{L}}_f(t)$ the set of points in $M_{\mathbf{R}} := \mathbf{R}^2$ or $\mathbf{T} \times \mathbf{R}$ which belong to an invariant curve $\gamma \in \overline{\mathcal{G}}_f$ such that $\gamma \subset M_{\mathbf{R}} \cap \{|r| < t\}$[20]

$$\overline{\mathcal{L}}_f(t) = \bigcup_{\substack{\gamma \in \overline{\mathcal{G}}_f \\ \gamma \subset M_{\mathbf{R}} \cap \{|r|<t\}}} \gamma.$$

We then define ($M_{\mathbf{R}} = \mathbf{R} \times \mathbf{R}$ or $\mathbf{T} \times \mathbf{R}$)

$$m_f(t) = \mathrm{Leb}_{M_{\mathbf{R}}}((M_{\mathbf{R}} \cap \{|r| < t\}) \setminus \overline{\mathcal{L}}_f(2t)).$$

Notation **1.1.** — *We shall use the following notations: if $a \geq 0$ and $b > 0$ are two real numbers we write $a \lesssim b$ for: "there exists a constant $C > 0$ independent of a and b such that $a \leq Cb$". If we want to insist on the fact that this constant C depends on a quantity β we write $a \lesssim_\beta b$. We shall also write $a \ll b$ to say that a/b is small enough and $a \ll_\beta b$ to express the fact that this smallness condition depends on β. The notations $b \gtrsim a$, $b \gtrsim_\beta a$, $b \gg a$ and $b \gg_\beta a$ are defined in the same way. When one has $a \lesssim b$ and $b \lesssim a$ we write $a \asymp b$.*

The 2-dimensional version of the KAM Theorem is the celebrated Moser's twist Theorem [32] (see also [43]):

Theorem (Moser). — Let f be a symplectic smooth diffeomorphism like (1.5) or (1.6) satisfying the twist condition (1.14). If f admits Birkhoff Normal Forms at the origin to all orders[21] *then, for any constant $a > 0$,*

$$m_f(t) \lesssim t^a. \tag{1.15}$$

Let us comment on the previous result. In the (CC) case, it is in fact enough to assume that ω in (1.5) is non-resonant since under this condition f admits Birkhoff Normal Forms to all orders. On the other hand, in the (AA) case the existence of the BNF to all orders, more precisely the existence of solutions to the related cohomological equations (see Lemma E.7), requires ω in (1.6) to be Diophantine; in the real analytic case a weaker arithmetic condition is enough,[22] see [44], [45]. If in the (CC) case ω is non-resonant only up to some low order, (1.15) holds only for some $a > 0$ (see [38]). If in the (AA) case we drop the assumption that ω is Diophantine (but we assume ω to be non-resonant) then, though no BNF is available, $m_f(t)$ in (1.15) goes to zero as t goes to zero but not necessarily as a power of t: indeed, in the corresponding (AA) case of sufficiently smooth Hamiltonian systems, Bounemoura proves in [6], in any number of degrees of freedom

[20] In the (AA) case $M_{\mathbf{R}} \cap \{|r| < t\} = \{(\theta, r) \in \mathbf{T} \times \mathbf{R},\ |r| < t\}$ and in the (CC) case $M_{\mathbf{R}} \cap \{|r| < t\} = \{(x, y) \in \mathbf{R} \times \mathbf{R},\ (1/2)(x^2 + y^2) < t\}$.

[21] This means one can define $B_N(f)$ for any $N \geq 2$.

[22] If we denote p_n/q_n the convergents of ω, it reads $\ln q_{n+1} = o(q_n)$. In comparison, the classical Diophantine condition amounts to $\ln q_{n+1} = O(\ln q_n)$.

and under a Kolmogorov non-degeneracy condition, that the origin is KAM stable[23] (see [13] for a previous similar result in two degrees of freedom and in the real analytic case) and provides measure estimates for the complement of the set of the invariant tori. Let us add that the twist condition (1.14) in Moser's Theorem can be considerably weakened (see for example [13], [12]). When $d = 1$, symplecticity (area preservation) can be replaced by the weaker *intersection property*.

When f is real analytic and ω (both in the (CC) and (AA) cases) is Diophantine one can get, by pushing to its limit the "standard" KAM method, a better estimate: for any $0 < \beta \ll 1$ and $t \ll_\beta 1$ one has

$$m_f(t) \lesssim \exp(-(1/t)^{\frac{1}{1+\tau(\omega)}-\beta}) \tag{1.16}$$

where we have defined for any irrational ω

$$\tau(\omega) = \limsup_{k\to\infty} \frac{-\ln \min_{l\in\mathbf{Z}} |k\omega - l|}{\ln k} = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{\ln q_n} \geq 1; \tag{1.17}$$

in the preceding formula $(p_n/q_n)_{n\geq 0}$ is the sequence of convergents[24] of ω. Note that if $\tau(\omega) < \infty$, then ω is Diophantine with exponent τ' for any $\tau' > \tau(\omega)$ (*cf.* (1.10)). In this case, the inequality (1.16) is known to be true with the exponent $1/(1+\tau')$ on the right hand side (see for example [26] and the references therein). If $\tau(\omega) = \infty$ we say that ω is *Liouvillian*.

1.3. *Optimal and improved measure estimates.* — The main results of our paper are that: (A) one can improve the exponent in (1.16) if BNF(f) converges; (B) in the "general case" the exponent in (1.16) is almost optimal. More precisely

Theorem **A.** — *Let f be a real analytic symplectic diffeomorphism $f : (\mathbf{R}\times\mathbf{R}, (0,0)) \circlearrowleft$ like (1.5) or $f : (\mathbf{T}\times\mathbf{R}, \mathcal{T}_0) \circlearrowleft$ like (1.6) satisfying the twist condition (1.14) and assume that in both cases ω is Diophantine. Then, if* BNF(f) *defines a converging series one has for any $0 < \beta \ll 1$ and $0 < t \ll_\beta 1$*

$$m_f(t) \lesssim \exp\left(-\left(\frac{1}{t}\right)^{(1/\tau(\omega))-\beta}\right). \tag{1.18}$$

On the other hand general real analytic twist symplectic diffeomorphisms like (1.5), (1.6) behave quite differently:

Theorem **B.** — *Let $\omega \in \mathbf{R}$ be Diophantine. There exist real analytic twist symplectic diffeomorphisms $f : (\mathbf{R}\times\mathbf{R}, (0,0)) \circlearrowleft$ like (1.5) or $f : (\mathbf{T}\times\mathbf{R}, \mathcal{T}_0) \circlearrowleft$ like (1.6) satisfying the twist*

[23] I.e. accumulated by a positive measure set of invariant quasi-periodic tori.

[24] As usual, if $\omega = 1/(a_1 + 1/(a_2 + 1/(\cdots)))$, $a_i \in \mathbf{N}^*$, we define $p_n/q_n = 1/(a_1 + 1/(a_2 + 1/(\cdots + 1/a_n)))$.

condition (1.14) and a sequence of positive numbers (t_k) *converging to zero such that for any* $0 < \beta \ll 1$, $0 < t_k \ll_\beta 1$

$$(\mathbf{1.19})\qquad m_f(t_k) \gtrsim \exp\left(-\left(\frac{1}{t_k}\right)^{(\frac{1}{1+\tau(\omega)})+\beta}\right).$$

As we already mentioned in the previous subsection, when ω is very Liouvillian, for example when $\liminf_n (q_n^{-1}\ln q_{n+1}) > 0$, it is not clear, in the (AA) case, how to define $\mathrm{BNF}(f)$. On the other hand, in the (CC) case, $\mathrm{BNF}(f)$ is defined whenever ω is non-resonant and as we will soon see in Theorem A' below, the result of Theorem A extends to this situation. One might still wonder whether a weaker Diophantine condition $\ln q_{n+1} = o(q_n)$ (or something slightly stronger) is enough to ensure the validity of Theorem A in the (AA) case (remember that in this case, $\mathrm{BNF}(f)$ is well defined). It seems possible that adapting Propositions 5.3, 5.5 to this situation (using e.g. [44], [45]) provides estimates that are still good enough to make the proof of Theorem A work.

Let us define for $\omega \in \mathbf{R}\smallsetminus\mathbf{Q}$

$$(\mathbf{1.20})\qquad t_n(\omega) = \frac{5\min(|b_2(f)|, |b_2(f)|^{-1})}{q_{n+1}q_n}.$$

Theorem **A'.** — *Let* ω *be Liouvillian and* $f : (\mathbf{R}\times\mathbf{R}, (0,0)) \circlearrowleft$ *be a real analytic symplectic diffeomorphism of the form (1.5) satisfying the twist condition (1.14). Then, if* $\mathrm{BNF}(f)$ *defines a converging series, one has for every* $k \in \mathbf{N}$ *large enough such that* $q_{k+1} \geq q_k^{10}$

$$(\mathbf{1.21})\qquad m_f(t_k(\omega)) \lesssim \exp(-q_{k+1}^{1/5}).$$

Note that if ω is Liouvillian, one has for infinitely many k, $q_{k+1} \geq q_k^{10}$.

On the other hand:

Theorem **B'.** — *For any* $\omega \in \mathbf{R}\smallsetminus\mathbf{Q}$, *there exist real analytic symplectic diffeomorphisms* $f : (\mathbf{R}\times\mathbf{R}, (0,0)) \circlearrowleft$ *of the form (1.5) satisfying the twist condition (1.14) such that for every* $\beta > 0$ *and infinitely many* $k \in \mathbf{N}$

$$(\mathbf{1.22})\qquad m_f(t_k(\omega)) \gtrsim \exp(-q_{k+1}^{\beta}).$$

Theorem A is a consequence of the following theorem:

Theorem **C** *(Small holes). — Let* ω *be Diophantine and let* $f : (\mathbf{R}\times\mathbf{R}, (0,0)) \circlearrowleft$ *of the form (1.5) or* $f : (\mathbf{T}\times\mathbf{R}, \mathcal{T}_0) \circlearrowleft$ *of the form (1.6) be a real analytic symplectic diffeomorphisms satisfying the twist condition (1.14). Then, for* $t > 0$, *there exists a finite collection* $\check{\mathcal{D}}_t$ *of pairwise disjoint disks* $\check{\mathrm{D}}$ *of the complex plane centered on the real axis such that, for any* $0 < \beta \ll 1$, $0 < t \ll_\beta 1$ *one has:*

(1) The number $\#\check{\mathcal{D}}_t$ *of disks in the collection* $\check{\mathcal{D}}_t$ *satisfies*

$$(\mathbf{1.23})\qquad \#\check{\mathcal{D}}_t \lesssim (1/t)^{1-\beta}$$

and one has

$$\forall\, \check{\mathrm{D}} \in \check{\mathcal{D}}_t \qquad |\check{\mathrm{D}} \cap \mathbf{R}| \lesssim \exp(-(1/t)^{\frac{1}{1+\tau(\omega)}-\beta}) \tag{1.24}$$

$$m_f(t) \lesssim \exp(-(1/t)^{(1/\tau(\omega))-\beta}) + \sum_{\check{\mathrm{D}} \in \check{\mathcal{D}}_t} |\check{\mathrm{D}} \cap \mathbf{R}|. \tag{1.25}$$

(2) If BNF(f) *converges, then for any* $t \ll_\beta 1$ *one has for each* $\check{\mathrm{D}} \in \check{\mathcal{D}}_t$

$$|\check{\mathrm{D}} \cap \mathbf{R}| \lesssim \exp(-(1/t)^{(1/\tau(\omega))-\beta}). \tag{1.26}$$

Estimate (1.24) explains why in general (without the assumption that BNF(f) converges) one only gets the estimate (1.16). We shall explain in Section 1.5.1 where these disks $\check{\mathrm{D}}$ come from.

There is a corresponding theorem in the Liouvillian (CC) case that implies Theorem A'. We shall not state it but we mention that it is a consequence of Theorem 12.6 and Corollary 13.6.

Theorems A, A', C are proved in Section 14 as consequences of Theorems 12.3, 12.6 and Corollaries 13.2, 13.6.

Theorems B and B' are consequences of Theorems E and E' which are stated in the next Section 1.4. These Theorems are proved in Section 16 which uses results from Section 15.

Because

$$1/\tau > 1/(1+\tau),$$

Theorems A–B, A'–B' clearly imply, in the (AA) and (CC) case, when $d = 1$, the *existence* of a diffeomorphism f of the form (1.5)–(1.6) with divergent BNF. We explain in the next Section 1.4, see Theorem D, that this implies Main Theorems 1–1' in the elliptic fixed point case and the action-angle case for any $d \geq 1$ and in a prevalent way.

1.4. *Prevalence of divergent BNF's.*

1.4.1. *The Dichotomy Theorem.* — Let us explain more precisely the dichotomy of R. Pérez-Marco mentioned in Section 1.1

Definition **1.2.** — *A subset* $\mathcal{A}$ *of a real affine space* $\mathcal{E}$ *is* (PM)*-prevalent*[25] *if there exists* $\mathrm{F}_0 \in \mathcal{A}$ *such that for any* $\mathrm{F} \in \mathcal{E}$ *the set* $\{t \in \mathbf{R},\ t\mathrm{F}_0 + (1-t)\mathrm{F} \notin \mathcal{A}\}$ *has 0 Lebesgue measure.*[26]

Pérez-Marco's dichotomy for Hamiltonians having a non-resonant elliptic fixed point can be reformulated the following way: let $\mathcal{E}_\omega$ be the affine space of real analytic

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[25] See [22] for the concept of prevalence.
[26] We can replace zero Lebesgue measure by zero (logarithmic) capacity like in Pérez-Marco's paper.

Hamiltonians

$$\mathrm{H}(x,y) = 2\pi \sum_{j=1}^{d'} \omega_j (x_j^2 + y_j^2)/2 + \mathrm{F}(x,y), \qquad \mathrm{F}(x,y) = \mathrm{O}^3(x,y)$$

which are perturbations of a given non-resonant quadratic part

$$\Omega_\omega(r) = 2\pi \langle \omega, r \rangle = 2\pi \sum_{j=1}^{d'} \omega_j (x_j^2 + y_j^2)/2$$

and let $\mathcal{A}_\omega$ be the set of those Hamiltonians which have a divergent BNF. Then, Pérez-Marco's dichotomy is: either for any $\mathrm{H} \in \mathcal{E}_\omega$, BNF(H) converges or $\mathcal{A}_\omega$ is (PM)-prevalent.

We now discuss the extension of Pérez-Marco's dichotomy to the case of symplectic diffeomorphisms in the (AA) and (CC)-cases.

Any real analytic symplectic diffeomorphism $f : (\mathbf{R}^d \times \mathbf{R}^d, (0,0)) \circlearrowleft$ of the form (1.5) or $f : (\mathbf{T}^d \times \mathbf{R}^d, \mathcal{T}_0) \circlearrowleft$ of the form (1.6) can be parametrized in the following convenient form:

$$f = \Phi_{2\pi\langle \omega, r\rangle} \circ f_{\mathrm{F}}, \tag{1.27}$$

where, $\mathrm{F} : (\mathbf{R}^d \times \mathbf{R}^d, (0,0)) \to \mathbf{R}$, $\mathrm{F} = \mathrm{O}^3(x,y)$ or $\mathrm{F} : (\mathbf{T}^d \times \mathbf{R}^d, \mathcal{T}_0) \to \mathbf{R}$, $\mathrm{F} = \mathrm{O}^2(r)$ is some real analytic function and where we denote $f_{\mathrm{F}} : (x,y) \mapsto (\widetilde{x}, \widetilde{y})$ or $(\theta, r) \mapsto (\widetilde{\theta}, \widetilde{r})$ the *exact-symplectic* map (see Section 4.5) defined implicitly by

$$\begin{cases} \widetilde{x} = x + \partial_{\widetilde{y}} \mathrm{F}(x, \widetilde{y}), \quad y = \widetilde{y} + \partial_x \mathrm{F}(x, \widetilde{y}) \quad \text{(CC case)} \\ \text{or} \\ \widetilde{\theta} = \theta + \partial_{\widetilde{r}} \mathrm{F}(\theta, \widetilde{r}), \quad r = \widetilde{r} + \partial_\theta \mathrm{F}(\theta, \widetilde{r}) \quad \text{(AA case).} \end{cases} \tag{1.28}$$

For $d \geq 1$, $\omega \in \mathbf{R}^d$ non-resonant, we define $\mathcal{S}_\omega(\mathbf{R}^d \times \mathbf{R}^d)$ (resp. $\mathcal{S}_\omega(\mathbf{T}^d \times \mathbf{R}^d)$) the set of real analytic symplectic diffeomorphisms $f : (\mathbf{R}^d \times \mathbf{R}^d, (0,0)) \circlearrowleft$ (resp. $f : (\mathbf{T}^d \times \mathbf{R}^d, \mathcal{T}_0) \circlearrowleft$) of the form $f = \Phi_{2\pi\langle \omega, r\rangle} \circ f_{\mathrm{F}}$ with $\mathrm{F} : (\mathbf{R}^d \times \mathbf{R}^d, (0,0)) \to \mathbf{R}$, $\mathrm{F} = \mathrm{O}^3(x,y)$ (resp. $\mathrm{F} : (\mathbf{T}^d \times \mathbf{R}^d, \mathcal{T}_0) \to \mathbf{R}$, $\mathrm{F} = \mathrm{O}^2(r)$) real analytic. We then say that a subset of $\mathrm{S}_\omega(\mathbf{R}^d \times \mathbf{R}^d)$ (resp. $\mathrm{S}_\omega(\mathbf{T}^d \times \mathbf{R}^d)$) is (PM)*-prevalent* if it is of the form $\{\Phi_{2\pi\langle \omega, r\rangle} \circ f_{\mathrm{F}},\ \mathrm{F} \in \mathcal{A}\}$ for some (PM)-prevalent subset $\mathcal{A}$ of $\mathrm{C}^\omega(\mathbf{R}^d \times \mathbf{R}^d, \mathbf{R}) \cap \mathrm{O}^3(x,y)$ (resp. $\mathrm{C}^\omega(\mathbf{T}^d \times \mathbf{R}^d, \mathbf{R}) \cap \mathrm{O}^2(r)$).

Here is the version of Pérez-Marco's Dichotomy Theorem [36] for real analytic symplectic diffeomorphisms of the $2d$-disk or the $2d$-cylinder.

Theorem **1.3** *(Dichotomy Theorem). — Let $d \geq 1$ and $\omega \in \mathbf{R}^d$ be a non-resonant frequency vector. Then, either for any $f \in \mathcal{S}_\omega(\mathbf{R}^d \times \mathbf{R}^d)$, the formal series* BNF(f) *converges (i.e. the series it defines has a positive radius of convergence), or there exists a* (PM)*-prevalent subset of $\mathcal{S}_\omega(\mathbf{R}^d \times \mathbf{R}^d)$ such that for any f in this subset* BNF(f) *diverges.*

The same dichotomy holds in $\mathcal{S}_\omega(\mathbf{T}^d \times \mathbf{R}^d)$ provided ω is Diophantine.

As we mentioned earlier Pérez-Marco's Dichotomy Theorem was proved in the setting of real analytic Hamiltonians having an elliptic fixed point. Its extension to the diffeomorphism setting follows essentially Pérez-Marco's arguments. We refer to Section 6.2 for further details in particular in the Action-Angle case (*cf.* Lemma 6.3).

1.4.2. *Prevalence of the divergence of the BNF: Main Theorems 1, 1'.* — As a Corollary of Theorem 1.3 we now obtain, using Theorems A and B, Theorems A' and B', the following precise formulation of Main Theorems 1, 1':

Theorem **D.** — *For any $d \geq 1$ and any non-resonant $\omega \in \mathbf{R}^d$, the set of $f \in \mathcal{S}_\omega(\mathbf{R}^d \times \mathbf{R}^d)$ with a divergent BNF is* (PM)*-prevalent. If ω is Diophantine the same result holds with $\mathcal{S}_\omega(\mathbf{T}^d \times \mathbf{R}^d)$ in place of $\mathcal{S}_\omega(\mathbf{R}^d \times \mathbf{R}^d)$.*

Proof. — We give the proof in the case of real analytic symplectic diffeomorphisms of the $2d$-disk.

Let $\omega = (\omega_1, \ldots, \omega_d) \in \mathbf{R}^d$ be non-resonant. According to Pérez-Marco's dichotomy (Theorem 1.3) it is enough to provide one example of a real analytic symplectic diffeomorphism of the $2d$-disk with diverging BNF and frequency vector ω at the origin to get the conclusion. Since ω is non-resonant, there exists $1 \leq j \leq d$ such that ω_j is irrational. According to whether ω_j is Diophantine or Liouvillian we use Theorems A and B or Theorems A' and B' to produce a real analytic symplectic diffeomorphism $f_j : (\mathbf{R}^2, 0) \circlearrowleft$ with frequency ω_j at the origin and with a divergent BNF. We now define $f : (\mathbf{R}^d \times \mathbf{R}^d, (0,0)) \circlearrowleft$ by $f(x_1, \ldots, x_d, y_1, \ldots, y_d) = (\widetilde{x}_1, \ldots \widetilde{x}_d, \widetilde{y}_1, \ldots, \widetilde{y}_d)$,

$$\begin{cases} \text{for } k \neq j, \ (\widetilde{x}_k + \sqrt{-1}\widetilde{y}_k) = e^{2\pi\sqrt{-1}\omega_k}(x_k + \sqrt{-1}y_k) \\ (\widetilde{x}_j, \widetilde{y}_j) = f_j(x_j, y_j). \end{cases}$$

This diffeomorphism is real analytic, symplectic and

$$\mathrm{BNF}(f)(r_1, \ldots, r_d) = \mathrm{BNF}(f_j)(r_j) + \sum_{k \in \{1,\ldots,d\} \smallsetminus j} 2\pi \omega_k r_k$$

is diverging since $\mathrm{BNF}(f_j)$ is. □

1.4.3. *Prevalence of optimal estimates: Main Theorem 2.* — We now present two theorems (Theorems E and E') stating that the measure estimates (1.19) of Theorem B and (1.22) of Theorem B' are prevalent. Together with Theorems A, A' and the fact that $1/(\tau + 1) < 1/\tau$, this gives a more precise meaning to our Main Theorem 2.

We shall treat the (AA) and (CC) cases separately.

Let $\mathcal{X}$ be the set $([-1,1]^2)^{\mathbf{N}^*} = \{(\zeta_{1,k}, \zeta_{2,k}) \in [-1,1]^2, \ k \in \mathbf{N}^*\}$ endowed with the product measure $\mu_\infty = (\mathrm{Leb}_{[-1,1]^2})^{\otimes \mathbf{N}^*}$.

(AA) Case. Let $f = \Phi_{2\pi\omega r} \circ f_{O(r^2)}$ be a real analytic symplectic twist map of the annulus of the form (1.6) and satisfying the twist condition (1.14).

For $\zeta \in \mathcal{X}$ and $h > 0$ we define $G_\zeta \in C^\omega(\mathbf{T} \times \mathbf{R})$ ($h > 0$ fixed)

$$G_\zeta(\theta, r) = r^{\overline{a}_3} \sum_{k \in \mathbf{N}^*} e^{-|k|h}(\zeta_{1,k}\cos(k\theta) + \zeta_{2,k}\sin(k\theta))$$

where $\overline{a}_3$ is some universal integer (appearing in Proposition G.1 of Appendix G).

Theorem **E** *((AA) case). — For any Diophantine ω and for any $0 < \beta \ll 1$, there exists an infinite set $\mathcal{N}_\beta \subset \mathbf{N}$ such that if $t_k = t_k(\omega)$ is the sequence defined by (1.20), then for μ_∞-almost $\zeta \in \mathcal{X}$, the estimate (1.19) of Theorem B with f replaced by f_ζ is satisfied for infinitely many $k \in \mathcal{N}_\beta$. In particular, using Theorem A, the BNF of $f_\zeta := f \circ f_{G_\zeta}$ is divergent for μ_∞-almost $\zeta \in \mathcal{X}$.*

(CC) Case. We formulate the corresponding result in the (CC) Case in a less general setting than in the (AA) Case. We assume that

$$f = \Phi_{\Omega((1/2)(x^2+y^2))} + O((x^2+y^2)^{\overline{a}_3})$$

where

$$(2\pi)^{-1}\Omega(r) = \omega r + b_2 r^2 + O(r^3), \qquad b_2 \neq 0.$$

We shall denote $\mathrm{sign}(b_2) = \pm 1$ if $\pm b_2 > 0$.

For $\zeta \in \mathcal{X}$, let G^*_ζ be the real analytic function

$$G^*_\zeta(x,y) = \left(\frac{x^2+y^2}{2}\right)^{\overline{a}_3} \sum_{k=1}^{\infty} \frac{\zeta_{1,k}}{2} \times \left(\left(\frac{x+iy}{\sqrt{2}}e^{-i\pi/4}\right)^{q_k} + \left(\frac{x-iy}{\sqrt{2}}e^{i\pi/4}\right)^{q_k}\right)$$
$$+ \frac{\zeta_{2,k}}{2i} \times \left(\left(\frac{x+iy}{\sqrt{2}}e^{-i\pi/4}\right)^{q_k} - \left(\frac{x-iy}{\sqrt{2}}e^{i\pi/4}\right)^{q_k}\right)$$

and

$$f_\zeta = f \circ \Phi_{G^*_\zeta}.$$

Theorem **E'** *((CC) Case). — For any non-resonant[27] (resp. Diophantine) ω and any $0 < \beta \ll 1$, there exist a non-empty set $s(\omega) \in \{-1, 1\}$ (resp. $s_\beta(\omega) \in \{-1, 1\}$) and an infinite set $\mathcal{N}' \subset \mathbf{N}$ (resp. $\mathcal{N}'_\beta \subset \mathbf{N}$) such that the following holds. If $\mathrm{sign}(b_2) \in s(\omega)$ (resp. $\mathrm{sign}(b_2) \in s_\beta(\omega)$), then, for μ_∞-almost $\zeta \in \mathcal{X}$, the estimate (1.22) of Theorem B' (resp. (1.19) of Theorem B) with f replaced by f_ζ is satisfied for infinitely many $k \in \mathcal{N}'$ (resp. $k \in \mathcal{N}'_\beta$). In particular, using Theorems A, A', for any non-resonant ω, the BNF of $f_\zeta := f \circ \Phi_{G^*_\zeta}$ is divergent for μ_∞-almost $\zeta \in \mathcal{X}$.*

We refer to Section 16 for the proof of Theorems E and E'.

[27] In the non-resonant case, the sets $s(\omega)$ and $\mathcal{N}' \subset \mathbf{N}$ do not depend on β.

1.5. *Some words on the proofs.* — The starting point of the proofs of Theorems A, A', and C is a KAM scheme that we implement on a holomorphic extension of the real analytic diffeomorphism f. This allows to work with holomorphic functions defined on complex domains "with holes" (i.e. disks which are removed). If these domains are "nice" we can use some quantitative form of the analytic continuation principle to propagate informations in the neighborhood of the origin, like the convergence of the BNF, to the neighborhoods of each hole. We illustrate this with the proof of Theorem C.

1.5.1. *Sketch of the proof of Theorem C.* — We describe it in the (AA) case. Let $f : (\mathbf{T}\times\mathbf{R}, \mathcal{T}_0) \circlearrowleft$, $\mathcal{T}_0 = \mathbf{T}\times\{0\}$, be a real analytic symplectic diffeomorphism of the form (1.27) with $b_2(f) \neq 0$ and ω Diophantine. After performing some steps of the Birkhoff Normal Form procedure mentioned in the introduction, we can assume that

$$f = \Phi_\Omega \circ f_{\mathrm{F}}, \quad (2\pi)^{-1}\Omega(r) = \omega r + b_2 r^2 + \cdots, \quad \mathrm{F} = \mathrm{O}(r^m) \tag{1.29}$$

where m is large enough and where f_{F} is the exact symplectic map (*cf.* (1.28)) associated to some real symmetric[28] holomorphic function $\mathrm{F} : \mathbf{T}_h \times \mathbf{D}(0,\overline{\rho}) \to \mathbf{C}$ $(h, \overline{\rho} > 0)$; the notations $\mathbf{T}_h$, $\mathbf{D}(0,\overline{\rho})$ are for $\mathbf{T}_h := ((\mathbf{R}+i]-h,h[)/(2\pi\mathbf{Z}))$, $\mathbf{D}(0,\overline{\rho}) = \{r\in\mathbf{C},\ |r| < \overline{\rho}\}$.

Adapted KAM Normal Form. — Theorem C can be seen as an improved version of the classic KAM Theorem on the positive Lebesgue measure of the set of points lying on invariant curves (*cf.* Moser's Theorem of Section 1.2). There are several ways to prove this standard KAM Theorem. A direct approach (which goes back to Arnold in his proof of Kolmogorov's theorem) is to find a sequence of (real symmetric) holomorphic symplectic diffeomorphisms g_i close to the identity, defined on smaller and smaller complex domains $\mathbf{T}_{h_i}\times \mathrm{U}_i$ $(h_{i-1} \geq h_i \geq h/2$, $\mathrm{U}_i \subset \mathrm{U}_{i-1} \subset \mathbf{D}(0,\overline{\rho}))$ and such that $g_i^{-1}\circ f \circ g_i$ gets closer and closer to some integrable[29] models Φ_{Ω_i}:

$$[\mathbf{T}_{h_i}\times \mathrm{U}_i] \qquad g_i^{-1}\circ f\circ g_i = \Phi_{\Omega_i}\circ f_{\mathrm{F}_i}, \qquad \|\mathrm{F}_i\| \ll 1 \tag{1.30}$$

(in the preceding formula, the set written on the left is a domain where the conjugation relation holds); see Figure 1. One then proves that g_i and Ω_i converge (in some sense) on $\mathbf{T}\times(\mathrm{U}_\infty\cap\mathbf{R})$ $(\mathrm{U}_\infty := \bigcap_i \mathrm{U}_i)$ to some limits g_∞, Ω_∞ and that $\mathrm{U}_\infty\cap\mathbf{R}$ (in general a Cantor set) has positive Lebesgue measure. The searched for set of f-invariant curves is then $\bigcup_{c\in\mathrm{U}_\infty\cap\mathbf{R}} g_\infty(\{r=c\})$ and one has for some fixed constant $a>0$ and any $\rho<\overline{\rho}$

$$m_f(\rho) \lesssim \|\mathrm{F}\|^a. \tag{1.31}$$

We refer to Theorem 12.1 for more details. The domains U_i can be chosen to be *holed domains i.e.* disks $\mathbf{D}(0,\rho_i)$ $(\rho_i \approx \overline{\rho})$ from which a finite number of small complex disks

[28] This means that it takes real values when θ and r are real.
[29] This means that Ω_i depends only on the r variable.

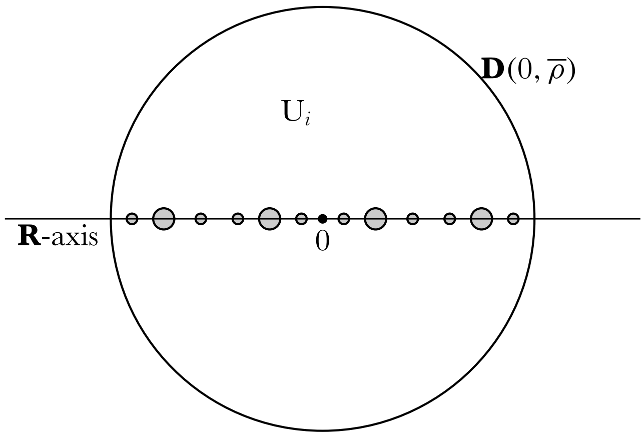

FIG. 1. — The holed domains U_i where the KAM-Normal Form U_i^{KAM} is defined (the holes, caused by resonances, are the grey disks)

centered on the real axis (the "holes" of U_i) have been removed. Removing these small disks is due to the necessity of avoiding *resonances* when one inductively construct g_i, Ω_i, F_i from g_{i-1}, Ω_{i-1}, F_{i-1}. More precisely, U_i is essentially obtained from U_{i-1} by removing "resonant disks" *i.e.* disks where the "frequency map" $(2\pi)^{-1}\partial\Omega_{i-1}$ is close to a rational number of the form l/k, $(l,k)\in\mathbf{Z}\times\mathbf{N}^*$, $\max(|l|,k)\lesssim N_{i-1}$ (N_i is an exponentially increasing (in i) sequence which is defined at the beginning of the inductive procedure). The sizes of the holes of U_i created by removing a finite number of disks from U_{i-1} decay very fast with i. We shall call a conjugation relation like (1.30) a(n) (approximate) *KAM Normal Form* for f. Its construction is presented in Section 7.

A useful observation (*cf.* Section 10) is that, depending on $\rho<\overline{\rho}$, one can choose indices $i_-(\rho)<i_+(\rho)$ such that all the holes D of the domain $U_{i_+(\rho)}$ that intersect $\mathbf{D}(0,\rho)$, are disjoint and are created at some step $i-1=i_D\in[i_-(\rho),i_+(\rho)]$ (hence $D\subset U_{i_D}$); moreover, $i_-(\rho)$ is large enough to ensure that the size of D is small. Writing (1.30) with $i=i_+(\rho)$ we get (note the change of notations)

$$(\mathbf{1.32})\qquad [\mathbf{T}_{h/2}\times U^{KAM}_{i_+(\rho)}]\qquad g^{-1}_{i_+(\rho)}\circ f\circ g_{i_+(\rho)}=\Phi_{\Omega^{KAM}_{i_+(\rho)}}\circ f_{F^{KAM}_{i_+(\rho)}},\qquad \|F^{KAM}_{i_+(\rho)}\|\ll 1.$$

This is what we call our *adapted KAM Normal Form* (adapted to $\mathbf{D}(0,\rho)$); see Section 10. With the choice we make for $i_+(\rho)$ we have

$$(\mathbf{1.33})\qquad \|F^{KAM}_{i_+(\rho)}\|\lesssim\exp(-(1/\rho)^{(1/\tau)-}),$$

where the last formula means: "for any $\beta>0$, $\|F^{KAM}_{i_+(\rho)}\|\lesssim_\beta\exp(-(1/\rho)^{(1/\tau)-\beta})$".

Hamilton-Jacobi Normal Forms. — *Cf.* Section 8. A hole $D\subset U_{i_D}$ of the domain $U_{i_+(\rho)}$ that is created at step i_D corresponds as we have mentioned to a resonance $(2\pi)^{-1}\partial\Omega^{KAM}_{i_D}\approx l/k$, $(l,k)\in\mathbf{Z}\times\mathbf{N}^*$, $\max(|l|,k)\lesssim N_{i_D}$ that appears when one constructs the KAM Normal Form (1.30) from step i_D to step i_D+1. In this resonant situation we are able to

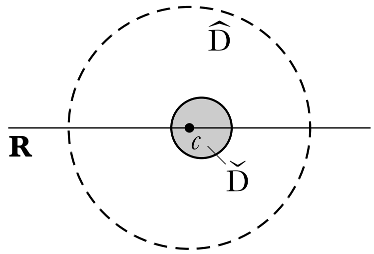

FIG. 2. — Hamilton-Jacobi Normal Form $\Omega_{\mathrm{D}}^{\mathrm{HJ}}$ close to a resonance $(2\pi)^{-1}\partial\Omega_{i_{\mathrm{D}}}^{\mathrm{KAM}}(c) = l/k$. The holomorphic function Ω^{HJ} is defined on the annulus $\widehat{\mathrm{D}} \setminus \check{\mathrm{D}}$

associate to D a *Hamilton-Jacobi Normal Form*, *cf.* Section 8, Proposition 8.1: there exists an annulus $\widehat{\mathrm{D}} \setminus \check{\mathrm{D}}$ ($\widehat{\mathrm{D}}, \check{\mathrm{D}}$ are disks), $\widehat{\mathrm{D}} \subset \mathrm{U}_{i_{\mathrm{D}}}$, $\widehat{\mathrm{D}} \supset \check{\mathrm{D}}$, $\widehat{\mathrm{D}} \supset \mathrm{D}$ ($\widehat{\mathrm{D}}$ is small but much bigger than D) on which one has

$$(\mathbf{1.34}) \qquad [\mathbf{T}_{h/9} \times \widehat{\mathrm{D}} \setminus \check{\mathrm{D}}] \qquad (g_{\mathrm{D}}^{\mathrm{HJ}})^{-1} \circ \Phi_{\Omega_{i_{\mathrm{D}}}} \circ f_{\mathrm{F}_{i_{\mathrm{D}}}} \circ (g_{\mathrm{D}}^{\mathrm{HJ}}) = \Phi_{\Omega_{\mathrm{D}}^{\mathrm{HJ}}} \circ f_{\mathrm{F}_{\mathrm{D}}^{\mathrm{HJ}}},$$

$$(\mathbf{1.35}) \qquad \|\mathrm{F}_{\mathrm{D}}^{\mathrm{HJ}}\| \lesssim \|\mathrm{F}_{i_+(\rho)}^{\mathrm{KAM}}\|.$$

See Figure 2. This HJ Normal Form also satisfies the important *Extension Property* which in some situation allows to bound above the size of $\check{\mathrm{D}}$ (note that in general the sizes of $\check{\mathrm{D}}$ and D are comparable). It states that if the holomorphic function $\Omega_{\mathrm{D}}^{\mathrm{HJ}}$, which is defined on the annulus $\widehat{\mathrm{D}} \setminus \check{\mathrm{D}}$, coincides to some very good order of approximation with a bounded holomorphic function defined on the *disk* $\widehat{\mathrm{D}}$, then $\check{\mathrm{D}}$ can be chosen to be small (see the quantitative statement of Proposition 8.1).

Proof of the first part (1.25) of Theorem C. — Applying the aforementioned standard KAM estimate (1.31) on the holed domain $\mathrm{U}_{i_+(\rho)}$ to $\Phi_{\Omega_{i_+(\rho)}} \circ f_{\mathrm{F}_{i_+(\rho)}^{\mathrm{KAM}}}$ (*cf.* (1.32)) and on each annulus $\widehat{\mathrm{D}} \setminus \check{\mathrm{D}}$ to $\Phi_{\Omega_{\mathrm{D}}^{\mathrm{HJ}}} \circ f_{\mathrm{F}_{\mathrm{D}}^{\mathrm{HJ}}}$, (*cf.* (1.34)) together with the estimate (1.35) we get that outside a set of measure $\sum_{\mathrm{D}\in\mathcal{D}_\rho} |\check{\mathrm{D}} \cap \mathbf{R}|$ the invariant curves of f cover a set the complement of which in $\mathbf{D}(0,\rho)$ has a measure $\lesssim \|\mathrm{F}_{i_+(\rho)}^{\mathrm{KAM}}\|^a$ for some $a > 0$; hence the inequality (1.25) by (1.33). For more details see the proof of Theorem 12.3.

Birkhoff Normal Forms. — *Cf.* Section 6. To prove the second part of Theorem C, (1.26) we need to introduce one further approximate Normal Form, namely the *approximate Birkhoff Normal Form* (*cf.* Section 6) valid on $\mathbf{T}_{h/2} \times \mathbf{D}(0, \rho^{b_\tau})$ ($b_\tau = \tau + 1$), $\mathbf{D}(0, \rho^{b_\tau}) \subset \mathrm{U}_{i_+(\rho)}^{\mathrm{KAM}}$

$$(\mathbf{1.36}) \qquad [\mathbf{T}_{h/2} \times \mathbf{D}(0, \rho^{b_\tau})] \qquad (g_\rho^{\mathrm{BNF}})^{-1} \circ f \circ (g_\rho^{\mathrm{BNF}}) = \Phi_{\Omega_\rho^{\mathrm{BNF}}} \circ f_{\mathrm{F}_\rho^{\mathrm{BNF}}},$$

$$(\mathbf{1.37}) \qquad \|\mathrm{F}_\rho^{\mathrm{BNF}}\| \lesssim \|\mathrm{F}_{i_+(\rho)}^{\mathrm{KAM}}\|.$$

See Figure 3.

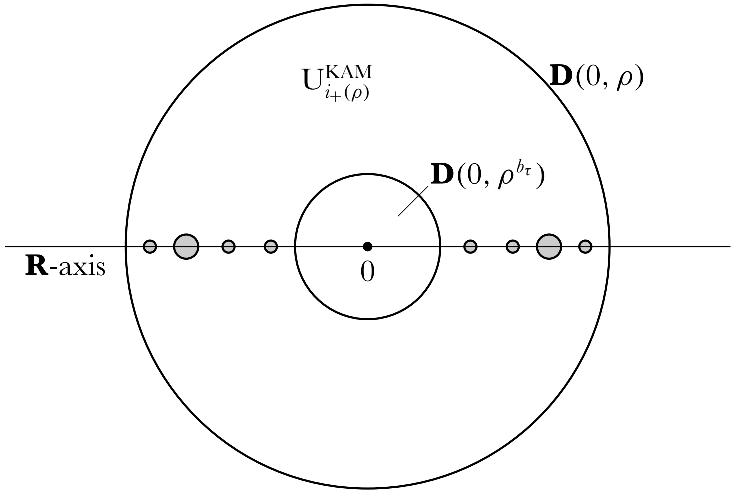

Fig. 3. — The approximate Birkhoff Normal Form Ω^{BNF}_ρ is defined on $\mathbf{D}(0, \rho^{b_\tau})$. It coexists with the KAM Normal Form $\Omega^{KAM}_{i_+(\rho)}$ defined on the holed domain $U_{i_+(\rho)}$

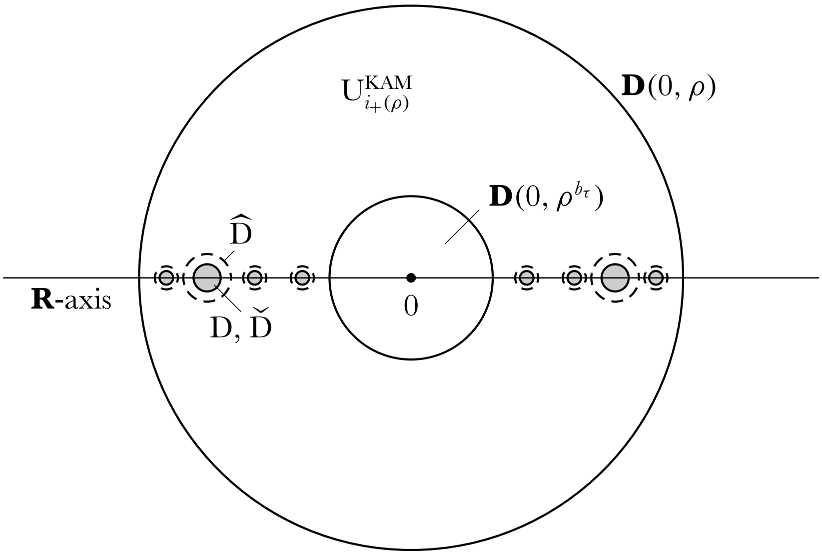

Fig. 4. — The three Normal Forms. The holomorphic function Ω^{BNF}_ρ is defined on $\mathbf{D}(0, \rho^{b_\tau})$; associated to each hole D the holomorphic function Ω^{HJ}_D is defined on the annulus $\widehat{D} \setminus \check{D}$. These Normal Form coincide with the KAM Normal Form $\Omega^{KAM}_{i_+(\rho)}$ defined on the holed domain $U^{KAM}_{i_+(\rho)}$

Proof of the second part (1.26) of Theorem C. — Having the three Normal Forms (1.32), (1.36), (1.34) in hand (see Figure 4) the proof of the second part of Theorem C relies on the following three principles.

– *Comparison Principle cf.* Section 9: since $F^{\rm KAM}_{i_+(\rho)}$, $F^{\rm HJ}_{\rm D}$ $F^{\rm BNF}_{\rho}$ are equally very small, all the previous Normal Forms almost coincide on the intersections of their respective domains of definition (this is done in Proposition 9.1); more precisely their *frequency maps* almost coincide

$$\Omega^{\rm BNF}_{\rho} \underset{\mathbf{D}(0,\rho^{b_\tau})\cap {\rm U}^{\rm KAM}_{i_+(\rho)}}{\simeq} \Omega^{\rm KAM}_{i_+(\rho)} \underset{{\rm U}^{\rm KAM}_{i_+(\rho)}\cap(\widehat{\rm D}\setminus\check{\rm D})}{\simeq} \Omega^{\rm HJ}_{\rm D} \tag{1.38}$$

where the symbol $a \underset{\rm V}{\simeq} b$ (a, b are functions and V is an open set) means here: for all $z\in{\rm V}$, $|a(z)-b(z)| \lesssim \exp(-(1/\rho)^{(1/\tau)-})$, ρ and τ being fixed. Moreover, if the formal BNF converges and equals a holomorphic function Ξ defined on, say, $\mathbf{D}(0,1)$, one has also (*cf.* Corollary 6.7)

$$\Xi \underset{\mathbf{D}(0,1)\cap\mathbf{D}(0,\rho^{b_\tau})}{\simeq} \Omega^{\rm BNF}_{\rho}$$

and in particular from (1.38) (we have $\mathbf{D}(0,\rho^{b_\tau})\subset {\rm U}^{\rm KAM}_{i_+(\rho)}$)

$$\Xi \underset{\mathbf{D}(0,1)\cap\mathbf{D}(0,\rho^{b_\tau})}{\simeq} \Omega^{\rm KAM}_{i_+(\rho)}. \tag{1.39}$$

– *No-Screening Principle, cf. Section 3:* Since Ξ and $\Omega^{\rm KAM}_{i_+(\rho)}$ almost coincide on $\mathbf{D}(0,\rho^{b_\tau})$ and are holomorphic on the bigger domain $\mathbf{D}(0,1)\cap{\rm U}^{\rm KAM}_{i_+(\rho)}$, one can be tempted to infer that they also almost coincide on this latter domain. A difficulty could appear here: an excessive number of holes of $\mathbf{D}(0,1)\cap{\rm U}^{\rm KAM}_{i_+(\rho)}$ (in comparison to their sizes) could cause some "screening effect" (like in Electrostatics) that prevents the propagation of the information given by (1.39) to "most of" the domain $\mathbf{D}(0,1)\cap{\rm U}^{\rm KAM}_{i_+(\rho)}$; see Section 3.2 for more details. This is the reason why, instead of working on the whole domain $\mathbf{D}(0,1)$ we work on the smaller one $\mathbf{D}(0,\rho)$. In this situation, the choice we make for $i_+(\rho)$ (*cf.* (10.301)) is such that the number of holes of $\mathbf{D}(0,1)\cap{\rm U}^{\rm KAM}_{i_+(\rho)}$ is not too big in comparison to their sizes; this is studied in Section 10, Proposition 10.4. This allows us to apply Proposition 3.1 and to extend the domain of validity of the approximate equality (1.39) to (a good part of) $\mathbf{D}(0,1)\cap{\rm U}^{\rm KAM}_{i_+(\rho)}$:

$$\Xi \underset{\mathbf{D}(0,1)\cap{\rm U}^{\rm KAM}_{i_+(\rho)}}{\simeq} \Omega^{\rm KAM}_{i_+(\rho)}. \tag{1.40}$$

– *Residue or Extension Principle cf. Section 8.8:* From (1.38), (1.40) one has

$$\Xi \underset{{\rm U}^{\rm KAM}_{i_+(\rho)}\cap(\widehat{\rm D}\setminus\check{\rm D})}{\simeq} \Omega^{\rm HJ}_{\rm D} \tag{1.41}$$

or, in other words, $\Omega^{\rm HJ}_{\rm D}$, which is defined on the *annulus* $\widehat{\rm D}\setminus\check{\rm D}$, coincides with a very good approximation with a *holomorphic function* defined on the *whole disk* $\widehat{\rm D}$. The aforementioned Extension Principle of Proposition 8.1, which essentially amounts to the computation of a residue (done in Paragraph 8.8.1), then tells us that the radius of $\check{\mathrm{D}}$ is much smaller than what we expected it to be: finally, $|\check{\mathrm{D}}\cap\mathbf{R}|\lesssim\exp(-(1/\rho)^{(1/\tau)-})$. This is (1.26).

For more details we refer to Proposition 10.7 and Corollary 13.2.

1.5.2. *On the proof of Theorem C in the elliptic fixed point case.* — The proof in the non-resonant elliptic fixed point case, $f:(\mathbf{R}^2,0)\circlearrowleft$, follows the same strategy especially if the frequency ω is Diophantine. A technical point is that to be able to implement the No-Screening Principle of Section 3 we need to work with domains $\mathrm{U}^{\mathrm{KAM}}_{i_+(\rho)}\supset\mathrm{U}^{\mathrm{BNF}}_{\rho}$ where $\mathrm{U}^{\mathrm{BNF}}_{\rho}$ is a disk around 0 (the estimate on the analytic capacity of this disk is then favorable). This is the reason why we cannot in this situation use Action-Angle variables since this would force us to work on angular sector domains and not disks.[30] Instead, we define our approximate BNF and KAM Normal Forms directly in Cartesian Coordinates. The formalism turns out to be the same as in the Action-Angle case (see Section 5), so we treat these two cases simultaneously. The case where ω is Liouvillian is done in a similar (and even simpler) way.

1.5.3. *On the proofs of Theorems B, B', E and E'.* — The proofs are more classical and based on the fact that, in the general case, resonances are associated to the existence of *hyperbolic* periodic points in the neighborhood of which no ("horizontal") invariant circle can exist. To see this in a special situation (we describe it in the (AA)-case) let $f=\Phi_\Omega\circ f_{\mathrm{F}}$, where $\mathbf{R}\ni r\mapsto(2\pi)^{-1}\Omega(r)=(p/q)r+br^2/2+\cdots\in\mathbf{R}$ with $p\in\mathbf{Z}$, $q\in\mathbf{N}^*$ mutually prime and, say, $b>0$ (for example $b=1$); we also assume that $\mathbf{T}\times\mathbf{R}\ni(\theta,r)\mapsto\mathrm{F}(\theta,r)=\mathrm{O}(r^3)\in\mathbf{R}$ and is $(2\pi/q)$*-periodic* in θ. The origin is thus *resonant* since $(2\pi)^{-1}\partial\Omega(0)=p/q$ is rational. One can *approximate* $\Phi_\Omega\circ f_{\mathrm{F}}$ by

$$f^{per}:=(\theta,r)\mapsto 2\pi(p/q,0)+\Phi^1_{\mathrm{H}}(\theta,r)$$

where Φ^1_{H} is the time-1 map of the Hamiltonian $\mathrm{H}(\theta,r)=br^2/2+\mathrm{F}(\theta,r)$. Observe that because we have assumed that $\mathrm{F}(\theta,r)$ is $2\pi/q$-periodic in θ, the same is true for Φ^1_{H}, hence the maps Φ^1_{H} and $(\theta,r)\mapsto(\theta,r)+2\pi(p/q,0)$ commute; thus, understanding the dynamics of $\widetilde{f}$ essentially amounts to understanding that of Φ^1_{H}. This latter dynamics is easy to analyze since it is the time-1 map of a Hamiltonian vector field in dimension 2, namely a *pendulum* on the cylinder $\mathbf{T}\times\mathbf{R}$. Indeed, if F is "typical", a change of coordinates leads us to the case where $\partial_\theta\mathrm{F}(0,0)=0$ and $\partial^2_\theta\mathrm{F}(0,0)<0$ hence $\mathrm{H}(\theta,r)=\mathrm{cst}+br^2/2-(1/2)|\partial^2_\theta\mathrm{F}(0,0)|\theta^2+h.o.t.$ Under this form it is clear that $(0,0)$ is a hyperbolic fixed point for Φ^1_{H} and since H is $2\pi/q$-periodic, the same is true for the points $(2\pi k/q,0)$, $k=0,\ldots,q-1$. Since these points are permuted by $(\theta,r)\mapsto(\theta,r)+2\pi(p/q,0)$, this shows that $(0,0)$ is a hyperbolic q-periodic point for $\widetilde{f}$ (this means a hyperbolic fixed point for $\widetilde{f}^q$). If $|\partial^2_\theta\mathrm{F}(0,0)|$ is not too small compared to the approximation $\|f-f^{per}\|$, the

[30] To say it shortly, in Poisson-Jensen's formula on subharmonic functions (see Section 3.1), the "weight" of a small disk $\mathbf{D}(0,\rho)\subset\mathbf{D}(0,1)$ is $1/|\ln\rho|$ while the "weight" of $\mathbf{D}(0,\rho)\cap\Delta\subset\Delta$, Δ being an angular sector at 0 is only ρ^a, $a>0$.

point (0,0) will also be a q-periodic hyperbolic point for f. However, a horizontal invariant circle cannot cross the stable or invariant manifolds of this periodic point; this establishes the existence of a zone in which horizontal invariant circles cannot pass. To quantify the size of this zone one just has to estimate the strength of the hyperbolicity of the periodic point and the size of the corresponding local stable and unstable manifolds.

The more general case where we do not assume *a priori* that $\mathrm{F}(\theta, r)$ is $2\pi/q$-periodic nor $\mathrm{F}(\theta, r) = \mathrm{O}(r^3)$ can essentially be reduced to the preceding example, provided F is small with respect to $1/q$. This requires the use of a resonant normal form described in Appendix G. For "generic" symplectic diffeomorphisms of the form (1.6) satisfying a twist condition (1.14) one can establish the existence of hyperbolic zones associated to any best rational approximation p_n/q_n of ω; these zones accumulate the origin. We refer to Sections 15 and 16 for more details.

1.6. *Organization of the paper.* — Section 2 is essentially dedicated to fixing some notations and introducing the notion of domains with holes that plays a central role in the KAM approach (à la Arnold). We discuss Cauchy's estimates and Whitney's extension Theorem in this framework. The not so standard notations used in the text are summarized in Section 2.6.

In Section 3 we give a brief account of what is the screening effect and we provide a no-screening criterion which will be useful for our purpose. It is based on Poisson-Jensen's formula on subharmonic functions applied in a domain with not too many holes (w.r.t. their sizes).

In Section 4 our main purpose is to check that estimates on compositions of generating functions hold in the case of domains with holes. We treat in a unified way the CC and AA cases. We also discuss invariant curves.

In Section 5 we study the (co)homological equations and state a proposition on the basic KAM step (Proposition 5.5).

Birkhoff Normal Forms (approximate and formal) are presented in Section 6 and Appendix E. We explain in Section 6.2 how Pérez-Marco's dichotomy extends to the diffeomorphism case.

Section 7 is dedicated to the KAM scheme which is central in our paper; we pay particular attention to the location of the holes of the KAM-domains.

In Section 8 we present the Hamilton-Jacobi Normal Form associated to each resonance appearing during the KAM scheme. Their construction is based on a Resonant Normal Form and an argument of approximation by vector fields the proofs of which are left in the Appendix, Sections G and H. The most important property of these Hamilton-Jacobi Normal Forms is the *Extension Property* that states that if the corresponding frequency map defined on a annulus is very close to a holomorphic function defined on a bigger disk containing the annulus, the domain of validity of this Normal Form is essentially this disk.

The Matching or Comparison Principle is presented in Section 9. It quantifies the fact that (exact) symplectic maps have essentially one frequency map.

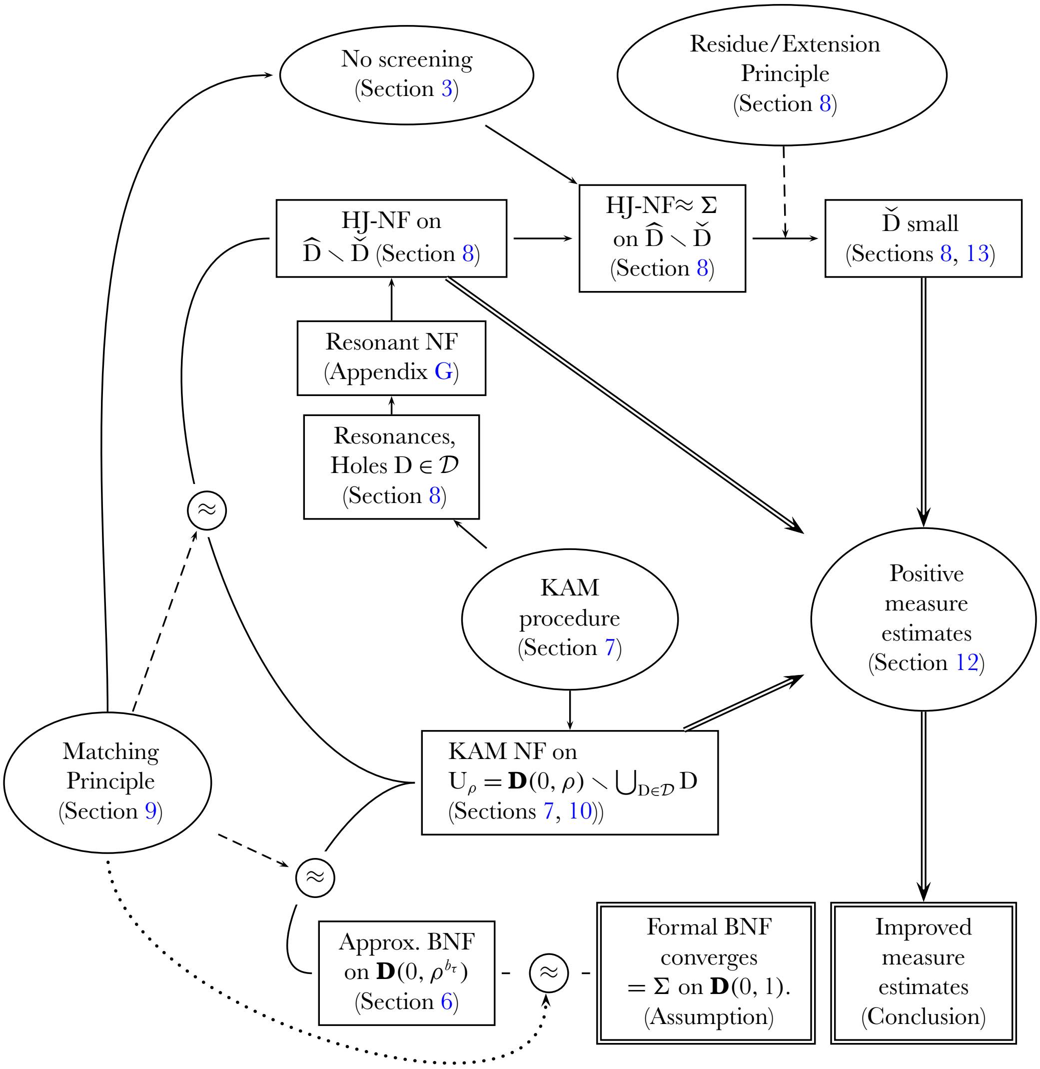

FIG. 5. — Plan of the proof of the improved measure estimate (1.26) of Theorem C

We construct in Section 10 and 11 our coexisting adapted KAM, BNF and HJ Normal Forms in the respective cases ω Diophantine or Liouvillian, the latter being easier to treat.

In Section 12 we first state a generalization of the classical KAM estimate on the measure of the set of invariant curves that hold on domains with holes (Theorem 12.1) and we apply it to our adapted KAM and HJ Normal forms to get measure estimates on the set of invariant curves lying in the union of the domains of definitions of these Normal Forms. This provides Theorems 12.3 and 12.6 which play an important role in the proofs of Theorems C, A and A'.

In Section 13 we use the Extension Principle of Section 8 to show that if the BNF converges the measure estimates provided by Theorems 12.3 and 12.6 improve considerably.

In Section 14 we conclude the proofs of Theorems C, A and A'.

The mechanism for the creation of zones of the phase space that do not intersect the set of invariant circles is presented in Section 15 (Proposition 15.1). This allows us to construct (prevalent) examples that satisfy Theorems B, B', E and E' in Section 16.

Finally, an Appendix completes the text by giving more details on the proofs of some statements or by presenting more or less classical methods that had to be adapted to our more specific situation.

2. Notations, preliminaries

Let $\mathbf{T}$ be the 1-dimensional torus $\mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z}) = \{x + 2\pi\mathbf{Z}, x \in \mathbf{R}\}$ and for $0 \leq h \leq \infty$

$$\mathbf{T}_h = \mathbf{T} \cup \{x + iy + (2\pi\mathbf{Z}),\ x, y \in \mathbf{R},\ |y| < h\} \qquad (i^2 = -1)$$

the complex cylinder of width $2h$. If $\theta_1 = (x_1 + iy_1) + (2\pi\mathbf{Z})$, $\theta_2 = x_2 + iy_2 + (2\pi\mathbf{Z}) \in \mathbf{T}_\infty$ we set $|\theta_1 - \theta_2|_{\mathbf{T}_\infty} := \min_{l\in\mathbf{Z}} |(x_1 - x_2 - 2\pi l) + i(y_1 - y_2)|$.

If $\rho > 0$ we denote by $\mathbf{D}(z, \rho) \subset \mathbf{C}$ the open disk of center z and radius ρ and by $\overline{\mathbf{D}}(z, \rho)$ its closure;[31] sometimes for short we shall write $\mathbf{D}_\rho$ for $\mathbf{D}(0, \rho)$ (and by $\overline{\mathbf{D}}_\rho$ its closure).

If $z = x + iy \in \mathbf{C}$, $(i = \sqrt{-1})$ $x, y \in \mathbf{R}$, (resp. $\theta = x + iy + (2\pi\mathbf{Z}) \in \mathbf{T}_\infty$), we denote by $\overline{z} = x - iy$ (resp. $\overline{\theta} = x - iy + (2\pi\mathbf{Z})$) its complex conjugate.

We define the *involutions* $\sigma_1, \sigma_2 : \mathbf{C}^2 \to \mathbf{C}^2$ and $\sigma_3 : \mathbf{T}_\infty \times \mathbf{C} \to \mathbf{T}_\infty \times \mathbf{C}$ by

$$(\mathbf{2.42}) \qquad \sigma_1(x, y) = (\overline{x}, \overline{y}), \qquad \sigma_2(z, w) = (i\overline{w}, i\overline{z}), \qquad \sigma_3(\theta, r) = (\overline{\theta}, \overline{r}).$$

For $w = (w_1, w_2)$, $w' = (w_1', w_2') \in \mathbf{C} \times \mathbf{C}$ (resp. $\in \mathbf{T}_\infty \times \mathbf{C}$) we define the distance $d(w, w') = \max(|w_1 - w_1'|, |w_2 - w_2'|)$ (resp. $d(w, w') = \max(|w_1 - w_1'|_{\mathbf{T}_\infty}, |w_2 - w_2'|)$).

If W is an open subset of $\mathbf{C} \times \mathbf{C}$ or of $\mathbf{T}_\infty \times \mathbf{C}$ and if $\mathrm{F} : \mathrm{W} \to \mathbf{C}$ we set

$$\|\mathrm{F}\|_{\mathrm{W}} = \sup_{\mathrm{W}} |\mathrm{F}|$$

(with the convention that $\|\mathrm{F}\|_{\mathrm{W}} = 0$ if W is empty). If a function $\mathrm{W} \ni (w_1, w_2) \mapsto \mathrm{F}(w_1, w_2)$ is differentiable enough (for the standard real differentiable structure on W) we can as usual define its partial derivatives[32] $\partial_{w_1}^{k_1} \partial_{\overline{w}_1}^{l_1} \partial_{w_2}^{k_2} \partial_{\overline{w}_2}^{l_2} \mathrm{F}$ $(k_1, k_2, l_1, l_2 \in \mathbf{N})$ and its

[31] With this notation $\overline{\mathbf{D}}(z, 0) = \emptyset$.

[32] Here we use the standard notation: if $w = t + is$, $(t, s) \in \mathbf{R}^2$, $\partial_w = (1/2)(\partial_t - i\partial_s)$ and $\partial_{\overline{w}} = \overline{\partial}_w = (1/2)(\partial_t + i\partial_s)$.

(total) j-th derivative $D^jF = (\partial_{w_1}^{k_1}\partial_{\overline{w}_1}^{l_1}\partial_{w_2}^{k_2}\partial_{\overline{w}_2}^{l_2}F)_{k_1+k_2+l_1+l_2=j}$ ($j \in \mathbf{N}$). We then define

$$\|D^jF\|_W = \max_{\substack{(k_1,l_1,k_2,l_2)\in\mathbf{N}^4\\ k_1+l_1+k_2+l_2=j}} \|\partial_{w_1}^{k_1}\partial_{\overline{w}_1}^{l_1}\partial_{w_2}^{k_2}\partial_{\overline{w}_2}^{l_2}F\|_W, \qquad \|F\|_{C^n(W)} = \max_{0\le j\le n}\|D^jF\|_W.$$

We denote by $C^n(W)$ the set of functions $F : W \to \mathbf{C}$ such that $\|F\|_{C^n(W)} < \infty$ and by $\mathcal{O}(W)$ the set of holomorphic functions $F : W \to \mathbf{C}$ (all the preceding partial derivatives of the form $\overline{\partial}_w = \partial_{\overline{w}}$ then vanish).

We say that an open set W of $M := \mathbf{C}^2$ or of $M := \mathbf{T}_\infty \times \mathbf{C}$ is σ_i*-symmetric* ($i = 1, 2, 3$) if it is invariant by σ_i ($\sigma_i(W) = W$); if W is σ_i-symmetric we say that a function $F : W \to \mathbf{C}$ is σ_i-symmetric if $F \circ \sigma_i = \overline{F}$ (the complex conjugate of F) and we denote by $C^n_{\sigma_i}(W)$, resp. $\mathcal{O}_{\sigma_i}(W)$, the set of C^n resp. holomorphic functions $F : W \to \mathbf{C}$ that are σ_i-symmetric. When no confusion is possible on the nature of the relevant σ_i involved, we shall often say σ-symmetric or even real symmetric instead of σ_i-symmetric. If W is σ-symmetric we use the notation $W_{\mathbf{R}} = \{w \in W,\ \sigma(w) = w\}$; if $W_{\mathbf{R}} \neq \emptyset$ then $F \in \mathcal{O}_\sigma(W)$ defines by restriction a map (still denoted by F) $F : W_{\mathbf{R}} \to \mathbf{R}$. Note that a function $F : (\mathbf{R}^2, 0) \to \mathbf{R}$ which is real analytic is in $\mathcal{O}_{\sigma_1}(\mathbf{D}(0,\rho) \times \mathbf{D}(0,\rho))$ for some $\rho > 0$.

Let W be a open set of $M := \mathbf{C}^2$ or $\mathbf{T}_\infty \times \mathbf{C}$. We denote by $\mathrm{Diff}^n(W)$, resp. $\mathrm{Diff}^{\mathcal{O}}(W)$, the set of C^n, resp. holomorphic, diffeomorphism $f : \widetilde{W} \to f(\widetilde{W}) \subset M$ defined on an open neighborhood $\widetilde{W}$ of W containing the closure of W.

Note that there exists a constant C depending only on M such that for any C^1-diffeomorphisms $f_1, f_2 : M \to M$ satisfying $\|f_1 - id\|_{C^1} \le 1$ one has

$$(\mathbf{2.43}) \qquad \|(f_2 \circ f_1) - id\|_{C^1} \le C(\|f_1 - id\|_{C^1} + \|f_2 - id\|_{C^1}).$$

If now W is a σ-symmetric open set of (M, σ) we denote by $\mathrm{Diff}^n_\sigma(W)$ resp. $\mathrm{Diff}^{\mathcal{O}}_\sigma(W)$ the set of $f \in \mathrm{Diff}^n(W)$, resp. $f \in \mathrm{Diff}^{\mathcal{O}}(W)$, such that $f \circ \sigma = \sigma \circ f$. It then defines by restriction a C^n, resp. real analytic, diffeomorphism (that we still denote f) $f : W_{\mathbf{R}} \to f(W_{\mathbf{R}}) \subset M_{\mathbf{R}}$.

When f, g are two σ-symmetric holomorphic diffeomorphisms we write

$$(\mathbf{2.44}) \qquad [W] \qquad f = g$$

(W possibly empty) to say that $f, g \in \mathrm{Diff}^{\mathcal{O}}_\sigma(W)$ coincide on an open neighborhood of W containing the closure of W.

2.1. *Domains* $W_{h,U}$. — Let $h \ge 0$ and U an open connected set of $\mathbf{C}$; we shall define domains $W^{AA}_{h,U}$ of $M = M^{AA} = \mathbf{T}_\infty \times \mathbf{C}$ (AA stands for "Action-Angle") and $W^{CC}_{h,U}$, W^{CC*}_h of $M = M^{CC} = M^{CC*} = \mathbf{C}^2$ (CC for "Cartesian Coordinates") the following way:

– *Cartesian Coordinates (*CC∗*)*: if $\rho_U := \sup\{|r|,\ r \in U\}$, the set $W^{CC*}_{h,U} \subset \mathbf{C} \times \mathbf{C}$ is

$$(\mathbf{2.45}) \qquad W^{CC*}_{h,U} = \{(x, y) \in \mathbf{C}^2,\ |x \pm iy| \le \sqrt{2}e^h\rho_U^{1/2},\ \frac{x^2 + y^2}{2} \in U\};$$

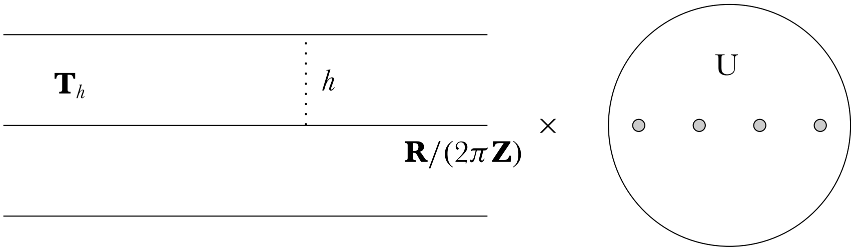

FIG. 6. — The domain $W^{AA}_{h,U} = \mathbf{T}_h \times U$

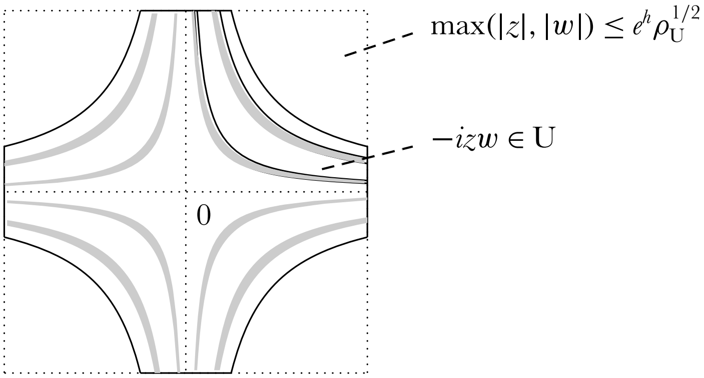

FIG. 7. — Schematic representation of the domain $W^{CC}_{h,U}$ (and $W^{CC*}_{h,U}$ if one makes the change of coordinates $z = \frac{1}{\sqrt{2}}(x+iy)$, $w = \frac{i}{\sqrt{2}}(x-iy)$)

– *Cartesian Coordinates (CC)*: if $\rho_U := \sup\{|r|,\ r \in U\}$, the set $W^{CC}_{h,U} \subset \mathbf{C} \times \mathbf{C}$ is

$$W^{CC}_{h,U} = \{(z,w) \in \mathbf{D}(0, e^h \rho_U^{1/2}) \times \mathbf{D}(0, e^h \rho_U^{1/2}),\ -izw \in U\}; \tag{2.46}$$

– *Action Angle coordinates (AA)*: the set $W^{AA}_{h,U}$ of $\mathbf{T}_\infty \times \mathbf{C}$ is

$$W^{AA}_{h,U} = \mathbf{T}_h \times U. \tag{2.47}$$

In all these three cases we denote by r the observable $(x,y) \mapsto (1/2)(x^2+y^2)$, $(z,w) \mapsto -izw$, $(\theta, r) \mapsto r$.

In Section 4.1 we shall see how one goes from (CC) (or (CC*)) to (AA) coordinates.

2.2. *Cauchy estimates.* — If $\delta > 0$ we denote by $\mathcal{U}_\delta(W) = \{w \in W,\ \mathbf{B}(w,\delta) \subset W\}$ (here $\mathbf{B}(w,\delta)$ is the ball $\{z \in M,\ d(z,w) < \delta\}$). Assume that $F \in \mathcal{O}(W)$. By differentiating (k_1+k_2)- times Cauchy complex integration formula

$$F(w_1,w_2) = \frac{1}{(2\pi i)^2} \int_{|w_1-\zeta_1|=\delta} \int_{|w_2-\zeta_2|=\delta} \frac{F(\zeta_1,\zeta_2)}{(w_1-\zeta_1)(w_2-\zeta_2)} d\zeta_1 d\zeta_2$$

one sees that if $\mathcal{U}_\delta(W)$ is not empty

$$\|\partial_{w_1}^{k_1}\partial_{w_2}^{k_2}F\|_{\mathcal{U}_\delta(W)} \le C_{k_1,k_2}\delta^{-(k_1+k_2)}\|F\|_W. \tag{2.48}$$

2.3. *Holed domains.*

2.3.1. *Holed domain of* $\mathbf{C}$. — A holed domain of $\mathbf{C}$ is an open set of $\mathbf{C}$ of the form

$$U = \mathbf{D}(c,\rho) \smallsetminus \bigcup_{i\in I} \overline{\mathbf{D}}(c_i,\rho_i), \tag{2.49}$$

for some $c \in \mathbf{C}$, $\rho > 0$, $c_i \in \mathbf{C}$, $\rho_i > 0$ and where I is a finite set which is either empty or such that for any $i \in I$, $\mathbf{D}(c_i,\rho_i) \cap \mathbf{D}(c,\rho) \neq \emptyset$. Note that the disks $\mathbf{D}(c_i,\rho_i)$ are not supposed to be included in $\mathbf{D}(0,\rho)$. It is not difficult to see that there exists a unique minimal $J_U \subset I$ (for the inclusion) such that $\bigcup_{i\in J_U}\overline{\mathbf{D}}(c_i,\rho_i) = \bigcup_{i\in I}\overline{\mathbf{D}}(c_i,\rho_i)$ and that the representation (2.49) with I replaced by J_U is then unique:

$$U = \mathbf{D}(c,\rho) \smallsetminus \bigcup_{i\in J_U} \overline{\mathbf{D}}(c_i,\rho_i). \tag{2.50}$$

We then denote by

$$\mathcal{D}(U) = \{\mathbf{D}(c_i,\rho_i),\ i \in J_U\}. \tag{2.51}$$

We shall call $\mathbf{D}(c,\rho)$ the *external disk* of U. We then set

$$\begin{cases} \rho_U := \overline{\mathrm{rad}}\,U := \rho, \quad \underline{\mathrm{rad}}(U) = \min_{i\in J_U}\rho_i \\ \underline{\mathrm{a}}(U) = (\sum_{i\in J_U}\rho_i^2)^{1/2} \\ \underline{\mathrm{card}}(U) = \#J_U \\ \underline{\mathrm{d}}(U) = \overline{\mathrm{rad}}(U) \text{ if } J_U = \emptyset, \quad \underline{\mathrm{d}}(U) = \min(\overline{\mathrm{rad}}(U), \underline{\mathrm{rad}}(U)) \text{ if } J_U \neq \emptyset. \end{cases} \tag{2.52}$$

If J_U is empty or if all the disks $\mathbf{D}(c_i,\rho_i)$, $i \in J_U$, are pairwise disjoint and included in $\mathbf{D}(c,\rho)$ we say that the holed domain U has disjoint holes and we call $\mathbf{D}(c_i,\rho_i)$ the holes of U (the bounded connected components of $\mathbf{C} \smallsetminus U$). We denote by $\mathcal{D}(U)$ the set of all these disks.

Note: We shall only consider in this paper holed domains (2.49) where the c_i are on the real axis.

2.3.2. *Holed domains of* $\mathbf{C} \times \mathbf{C}$ *or* $\mathbf{T}_\infty \times \mathbf{C}$. — These are by definition sets of the form $W_{h,U}$ where $h > 0$ and U is a holed domain; see (2.46) or (2.47). We then define

$$\underline{\mathrm{d}}(W_{h,U}) = \min(h, \underline{\mathrm{d}}(U)).$$

2.3.3. *Deflation of a holed domain.* — If $\delta\in\mathbf{R}$ we use the notation $e^{-\delta}\mathbf{D}(c,\rho)$ for

$$e^{-\delta}\mathbf{D}(c,\rho)=\mathbf{D}(c,e^{-\delta}\rho).$$

If $\mathrm{U}\subset\mathbf{C}$ is a holed domain of the form (2.50) and if $\delta>0$ we denote by $e^{-\delta}\mathrm{U}\subset\mathrm{U}$ the (possibly empty) open set

$$e^{-\delta}\mathrm{U}=\mathbf{D}(c,e^{-\delta}\rho)\smallsetminus\bigcup_{i\in \mathrm{J_U}}\overline{\mathbf{D}}(c_i,e^{\delta}\rho_i).$$

Similarly if $0<\delta<h$

$$e^{-\delta}\mathrm{W}_{h,\mathrm{U}}=\mathrm{W}_{h-\delta/2,e^{-\delta}\mathrm{U}}.$$

We make the following simple observations (the first two items are proved by area considerations):

Lemma **2.1.** — *For* $1>\delta>0$ *one has:*
(1) For any $z\in\mathbf{D}(c,\rho)$, $\operatorname{dist}(z,\mathrm{U})\leq 2\,\underline{a}(\mathrm{U})$.
(2) If $\rho^2>2e^{4\delta}\sum_{i\in\mathrm{J_U}}\rho_i^2$ *then* $e^{-\delta}\mathrm{U}$ *is not empty.*
(3) If $e^{-\delta}\mathrm{U}$ *is not empty, then for any* $z\in e^{-\delta}\mathrm{U}$ *one has*

$$\mathbf{D}(z,(1/2)\delta\,\underline{d}(\mathrm{U}))\subset\mathrm{U}.$$

2.3.4. *Reformulation of Cauchy's Inequalities.* — Using item 3 of Lemma 2.1 we can in particular reformulate inequalities (2.48) when W is of the form $\mathrm{W}_{h,\mathrm{U}}$ and $\mathrm{F}\in\mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$:

$$\|\mathrm{D}^m\mathrm{F}\|_{e^{-\delta}\mathrm{W}_{h,\mathrm{U}}}\leq \mathrm{C}_m\delta^{-m}\,\underline{\mathrm{d}}(\mathrm{W}_{h,\mathrm{U}})^{-m}\|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}}. \tag{2.53}$$

One can sometimes obtain better estimates.

– In the (AA)-case, if $0<\delta<h$, one has

$$\|\partial_\theta^k\mathrm{F}\|_{e^{-\delta}\mathrm{W}_{h,\mathrm{U}}}\lesssim\delta^{-k}\|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}} \tag{2.54}$$

– In the (CC)-case, if $\mathrm{U}=\mathbf{D}(0,\rho)$ and $\delta<1/2$ one has $e^{-\delta}\mathrm{W}_{h,\mathbf{D}(0,\rho)}\subset\mathcal{U}_{\widetilde{\delta}}(\mathrm{W}_{h,\mathbf{D}(0,\rho)})$ with $\widetilde{\delta}=\rho^{1/2}e^{-h}\delta/4$ and thus

$$\|\nabla\mathrm{F}\|_{e^{-\delta}\mathrm{W}_{h,\mathrm{U}}}\lesssim e^h\delta^{-1}\rho^{-1/2}\|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}}. \tag{2.55}$$

2.4. *Whitney type extensions on domains with holes.* — The discussion that follows will be useful in the construction of the KAM Normal Form of Section 7.

Let U be a real symmetric holed domain

$$\mathrm{U}=\mathbf{D}(0,\rho)\smallsetminus\bigcup_{i\in\mathrm{J_U}}\overline{\mathbf{D}}(c_i,\rho_i),\qquad c_i\in\mathbf{R}, \tag{2.56}$$

$h > 0$, $W_{h,U}$ one of the domains defined in Section 2.1 and $F : W_{h,U} \to \mathbf{C}$ be a C^k[33] σ-symmetric function i.e. $F \circ \sigma = \overline{F}$ (the complex conjugate of F). We say that a C^k, σ-symmetric function[34] $F^{Wh} : W_{h,\mathbf{C}} \to \mathbf{C}$ is a *Whitney extension*[35] for $(F, W_{h,U})$ if

$$\forall\, m \in W_{h,U}, \qquad F^{Wh}(m) = F(m).$$

Note that since U is open this implies that for all $0 \le j \le k$, $D^j F$ and $D^j F^{Wh}$ coincide on $W_{h,U}$.

We shall construct such Whitney's extensions in two situations.

Lemma **2.2.** — *Let* $F \in \mathcal{O}_\sigma(W_{h,U})$. *For any* $\delta \in]0, 1[$, *there exists a* C^k, σ*-symmetric function* $F^{Wh} : W_{h,\mathbf{C}} \to \mathbf{C}$ *such that*

$$\forall\, m \in e^{-\delta} W_{h,U}, \qquad F^{Wh}(m) = F(m) \tag{2.57}$$

$$\sup_{0\le j\le k} \|D^j F^{Wh}\|_{W_{h,\mathbf{C}}} \le C(1 + \#J_U)^k (\delta\, \underline{d}(U))^{-2k} \max_{0\le j\le k} \|D^j F\|_{W_{h,e^{-\delta/10}U}}. \tag{2.58}$$

Proof. — See Section B.1 of the Appendix. □

Notation **2.3.** — *We denote by* $\widetilde{\mathcal{O}}_\sigma(W_{h,U})$ *the set of* C^3, σ*-symmetric maps* $F : W_{h,\mathbf{C}} \to \mathbf{C}$ *such that the restriction of* F *on* $W_{h,U}$ *is holomorphic.*

Definition **2.4.** — *Let* $A \ge 1$, $B \ge 1$, $U \subset \mathbf{C}$ *a* σ*-symmetric holed domain. We say that a* σ*-symmetric* C^3 *function* $\Omega : U \to \mathbf{C}$ *satisfies an* (A, B)*-twist condition on* U *if*

$$\forall\, r \in U \cap \mathbf{R},\ A^{-1} \le \frac{1}{2\pi}\partial^2\Omega(r) \le A, \quad \text{and} \quad \|\frac{1}{2\pi}D^3\Omega\|_{\mathbf{C}} \le B. \tag{2.59}$$

If U is a disk $\mathbf{D}(0, \rho_0)$ one can construct for some $0 < \overline{\rho} < \rho_0$ a C^3, σ-symmetric Whitney extension for Ω on $\mathbf{D}(0, \overline{\rho})$ that satisfies an (A, B)-twist condition on $\mathbf{D}(0, \overline{\rho})$.

Lemma **2.5.** — *Let* $\Omega \in \mathcal{O}_\sigma(\mathbf{D}(0, \rho_0))$ $(\rho_0 \le 2)$

$$(2\pi)^{-1}\Omega(z) = \omega_0 z + b_2 z^2 + O(z^3), \qquad \|\Omega\|_{\mathbf{D}(0,\rho_0)} \le 1, \qquad \text{with } b_2 > 0.$$

There exists $0 < \overline{\rho} < \rho_0$, $B \ge 0$ *and a* C^3, *real symmetric extension* $\Omega^{Wh} \in \widetilde{\mathcal{O}}_\sigma(\mathbf{C})$ *of* $(\Omega, \mathbf{D}(0, \overline{\rho}))$ *that satisfies an* (A, B)*-twist condition on* $\mathbf{C}$ *with* $A = 3\max(b_2, b_2^{-1})$.

Proof. — See Appendix B.2. □

Notation **2.6.** — *We denote by* $\mathcal{TC}(A, B)$ *the set of* C^3, *real symmetric maps* $\Omega : \mathbf{C} \to \mathbf{C}$ *satisfying an* (A, B)*-twist condition (2.59) with* $U = \mathbf{C}$.

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[33] Differentiability here is related to the real differentiable structure of $W_{h,\mathbf{C}}$.
[34] The exponent Wh stands for "Whitney".
[35] See [55], [50].

Let $\mathrm{U}\subset\mathbf{C}$ be a σ-symmetric connected holed domain as in (2.56).

Proposition **2.7.** — *If* $\Omega\in\widetilde{\mathcal{O}}_\sigma(\mathrm{U})\cap\mathcal{TC}(\mathrm{A},\mathrm{B})$ *with*

$$8\times\max(\rho,\underline{a}(\mathrm{U}))\times\mathrm{A}\times\mathrm{B}<1 \tag{2.60}$$

then the following holds. For any $\nu\in]0,(6\mathrm{A}^2\mathrm{B})^{-1}[$ *and any* $\beta\in\mathbf{R}$, *either for any* $z\in\mathrm{U}$

$$|\omega(z)-\beta|\geq\nu \qquad (\omega=(2\pi)^{-1}\partial\Omega) \tag{2.61}$$

or there exists a unique $c_\beta\in]-\rho-2\mathrm{A}\nu,\rho+2\mathrm{A}\nu[$ *such that* $\omega(c_\beta)=\beta$ *and for any* $z\in\mathrm{U}\smallsetminus\overline{\mathbf{D}}(c_\beta,3\mathrm{A}\nu)$ *one has*

$$|\omega(z)-\beta|>\nu.$$

Proof. — See Appendix B.3. □

2.5. *Notation* $\mathfrak{O}_p$. — Let $h>0$, U be a holed domain, functions $\mathrm{F}_1,\ldots,\mathrm{F}_n\in\mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$ and $l\in\mathbf{N}^*$. We define the relation

$$\mathrm{G}=\mathfrak{O}_l(\mathrm{F}_1,\ldots,\mathrm{F}_n)$$

as follows: there exist $a\in\mathbf{N}^*$, $\mathrm{C}>0$ and $\mathrm{Q}(\mathrm{X}_1,\ldots,\mathrm{X}_n)$ a homogeneous polynomial (independent of U) of degree l in the variables $(\mathrm{X}_1,\ldots,\mathrm{X}_n)$ such that for any $0<\delta<h/2$ satisfying

$$\mathrm{C}\,\underline{\mathrm{d}}(\mathrm{W}_{h,\mathrm{U}})^{-a}\delta^{-a}\max_{1\leq i\leq n}\|\mathrm{F}_i\|_{\mathrm{W}_{h,\mathrm{U}}}\leq 1 \tag{2.62}$$

one has $\mathrm{G}\in\mathcal{O}(e^{-\delta}\mathrm{W}_{h,\mathrm{U}})$ and

$$\|\mathrm{G}\|_{e^{-\delta}\mathrm{W}_{h,\mathrm{U}}}\leq\underline{\mathrm{d}}(\mathrm{W}_{h,\mathrm{U}})^{-a}\delta^{-a}\mathrm{Q}(\|\mathrm{F}_1\|_{\mathrm{W}_{h,\mathrm{U}}},\ldots,\|\mathrm{F}_n\|_{\mathrm{W}_{h,\mathrm{U}}}). \tag{2.63}$$

We shall use the notation $\dot{\mathfrak{O}}_l(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ if the polynomial Q is null when $\mathrm{X}_1=0$ *i.e.* $\mathrm{Q}(0,\mathrm{X}_2,\ldots,\mathrm{X}_n)=0$; for example if $l=n=2$, $\mathrm{Q}(\mathrm{X}_1,\mathrm{X}_2)=\mathrm{X}_1\mathrm{X}_2+\mathrm{X}_1^2$.

When we want to keep track of the exponent a appearing in (2.62), (2.63) we shall use the symbol $\mathfrak{O}_l^{(a)}$.

When δ satisfies (2.62) we write

$$\delta=\mathfrak{d}^{a,\mathrm{C}}(\mathrm{F}_1,\ldots,\mathrm{F}_n;\mathrm{W}_{h,\mathrm{U}}) \tag{2.64}$$

and we use the short hand notation

$$\delta=\mathfrak{d}(\mathrm{F}_1,\ldots,\mathrm{F}_n;\mathrm{W}_{h,\mathrm{U}}) \tag{2.65}$$

to say that (2.64) holds for some positive constants a, C large enough and independent of $\mathrm{F}_1,\ldots,\mathrm{F}_n$, $\underline{\mathrm{d}}(\mathrm{W}_{h,\mathrm{U}})$.

Remark **2.1.** — Note that if $U = \mathbf{D}(0, \rho_0)$ is a disk containing 0 and $F \in \mathcal{O}(W^*_{h,U})$, $* = CC, AA$, one has

$$F(z, w) = O^p(z, w) \iff \forall\, 0 \le \rho \le \rho_0,\ \|F\|_{W_{h,\mathbf{D}(0,\rho)}} \lesssim \rho^{p/2}$$

$$F(\theta, r) = O^p(r) \iff \forall\, 0 \le \rho \le \rho_0,\ \|F\|_{W_{h,\mathbf{D}(0,\rho)}} \lesssim \rho^{p}$$

hence if $F_1, \ldots, F_n \in \mathcal{O}(W^{CC}_{h,U})$ (resp. $\in \mathcal{O}(W^{AA}_{h,U})$) satisfy $F_i(z, w) = O^p(z, w)$ (resp. $F_i = O^p(r)$), $1 \le i \le n$, one has

$$\mathfrak{D}_m(F_1, \ldots, F_n) = O^{mp-2a}(z, w) \qquad (\text{resp. } O^{mp-a}(r)).$$

2.6. *Summary of the various notations used in the text.*

- $a \lesssim b$, $a \lesssim_\beta b$, $a \ll b$, $a \ll_\beta b$, $a \asymp b$ *etc.* See Notation 1.1.
- $a \lesssim \exp(b-)$ means: for all $\beta > 0$, one has $a \lesssim_\beta \exp(b - \beta)$.
- $\mathbf{A}(z; \lambda_1, \lambda_2) = \{w \in \mathbf{C},\ \lambda_1 < |w - z| < \lambda_2\}$.
- $\mathbf{T}_h = \{x + iy + (2\pi\mathbf{Z}),\ x, y \in \mathbf{R},\ |y| < h\}$ $(i^2 = -1)$. $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$.
- ρ_U, $\underline{d}(U)$, $\underline{a}(U)$, J_U, $\mathcal{D}_U$: see Section 2.3.1.
- $W^{AA}_{h,U} = \mathbf{T}_h \times U$, $W^{CC}_{h,U} = \{(z, w) \in \mathbf{D}(0, e^h\rho_U^{1/2}) \times \mathbf{D}(0, e^h\rho_U^{1/2}),\ -izw \in U\}$.
- $e^{-\delta}W_{h,U} = W_{h-\delta/2, e^{-\delta}U}$.
- $\mathcal{O}_\sigma(W)$: the set of σ-symmetric holomorphic function on W.
- $\widetilde{\mathcal{O}}_\sigma(W_{h,U})$: see Notation 2.3. $\widetilde{\mathrm{Symp}}_\sigma(W_{h,U})$: see Notation 4.8.
- $\delta = \mathfrak{d}^{a,C}(F; W_{h,U})$, $\delta = \mathfrak{d}(F; W_{h,U})$, $\mathfrak{D}_l(F, G)$, $\mathfrak{D}^a_l(F, G)$: Section 2.5.
- $\mathcal{TC}(A, B)$, (A, B)-twist condition: see Notation 2.6 and Definition 2.4.
- $\mathcal{G}(f, W)$, $\mathcal{L}(f, W)$: see Notation 4.1.
- $\Phi_F = \phi^1_{J\nabla F}$. For the canonical map f_F see (4.87) and (4.88).
- $[\Omega] \cdot Y = Y \circ \Phi_\Omega - Y$. See Section 4.7.
- $\mathcal{M}_n(F)$, $T_N F$, $R_N F$: see Section 5.1.
- $A \triangle B = (A \cup B) \setminus (A \cap B)$.

3. A no-screening criterion on domains with holes

3.1. *Harmonic measures.* — Let U be a bounded open set of the complex plane with boundary ∂U. We can define its *Green function*, $g_U : U \times U \to \mathbf{R}$ as follows: for any $z \in U$, $-g(z, \cdot)$ is the function equal to 0 on the boundary ∂U of U, which is subharmonic on U, harmonic on $U \setminus \{z\}$ and which behaves like $\log|z - w|$ when $w \in U$ goes to z (this means that $g(z, w) + \log|z - w|$ stays bounded when w goes to z). The Green function g_U is thus nonnegative. We denote by $\omega_U : U \times \mathrm{B}or(\partial U) \to [0, 1]$ the *harmonic measure* of U (here $\mathrm{B}or(\partial U)$ is the set of borelian subsets of ∂U) defined as follows: if $z \in U$ and $I \in \mathrm{B}or(\partial U)$ (one can assume I is an arc for example if ∂U is a union of circles) then the function $\omega_U(\cdot, I)$ is the unique harmonic function defined on U, having a continuous extension to $\overline{U}$ and such that $\omega_U(z, I) = 1$ if $z \in I$ and 0 if $z \in \partial U \setminus I$. *Poisson-Jensen formula* (*cf.* [39]) asserts that for any subharmonic function $u : U \to \mathbf{C}$

$$u(z) = \int_{\partial U} u(w) d\omega_U(z, w) - \int_U g_U(z, w) \Delta u(w)$$

where Δu is the usual Laplacian of u. In particular, if f is a holomorphic function on U, the application of this formula to $u(z) = \ln |f(z)|$ gives

$$\ln |f(z)| = \int_{\partial U} \ln |f(w)| d\omega_U(z, w) - \sum_{w:f(w)=0} g_U(z, w)$$

and thus since g_U is nonnegative

$$\ln |f(z)| \leq \int_{\partial U} \ln |f(w)| d\omega_U(z, w). \tag{3.66}$$

Though we shall not use it in this paper we mention the fact that the harmonic measure $\omega_U(z, \cdot)$ can also be defined in a probabilistic way by using Brownian motions: if $W_z(t)$ is the value at time t of a Brownian motion issued from the point z (at time 0) and $T_{z,I}$ is the stopping time adapted to the filtration $\mathcal{F}_z$ of hitting I before $\partial U \setminus I$, then $\omega_U(z, I) = \mathbf{E}(\mathbf{1}_I(W_z(T_{z,I})))$; hence

$$\ln |f(z)| \leq \mathbf{E}(\ln |f(W_z(T_{z,I}))|). \tag{3.67}$$

This probabilistic interpretation is often useful in trying to get a hunch of the behavior of the harmonic measures.

3.2. *Screening effect.* — Assume now that $|f| \leq 1$ on U and that $|f| \ll 1$ on some nonempty open subset $B \subset U$. Does this imply that f is small on "most" of U? Formula (3.66) applied to the domain $U \setminus \overline{B}$ in place of U yields for any $z \in U \setminus \overline{B}$

$$\ln |f(z)| \leq \omega_{U \setminus \overline{B}}(z, \partial B) \times \ln \|f\|_B \tag{3.68}$$

and answering the preceding question amounts to getting good estimates from below on the nonnegative function $\omega_{U \setminus \overline{B}}(z, \partial B)$.

For example take $U = \mathbf{D}(0, 1)$ and $B = \mathbf{D}(0, \sigma)$ ($\partial(U \setminus \overline{B})$ is then the union of the two circles of center 0 and radii σ and 1). It is easy to see that for $|z| \leq 1/2$

$$\omega_{U \setminus \overline{B}}(z, \partial B) = \ln |z| / \ln \sigma \geq \ln(1/2) / \ln \sigma \tag{3.69}$$

and the preceding formula (3.68) applied to the domain $U \setminus \overline{B}$ shows that

$$\ln \|f\|_{\mathbf{D}(0,1/2)} \lesssim \frac{1}{|\ln \sigma|} \ln \|f\|_{\mathbf{D}(0,\sigma)}. \tag{3.70}$$

If we assume for instance

$$\|f\|_{\mathrm{B}} \leq e^{-\mathrm{N}^{\alpha}}, \qquad e^{-\mathrm{N}^{\beta}} \leq \sigma \leq 1/10, \qquad 0 < \beta < \alpha, \qquad \mathrm{N} \gg 1$$

this yields

$$\omega_{\mathrm{U}\setminus\overline{\mathbf{D}(0,\sigma)}}(z, \partial\mathbf{D}(0,\sigma)) \gtrsim \mathrm{N}^{-\beta} \tag{3.71}$$

$$\ln \|f\|_{\mathbf{D}(0,1/2)} \lesssim -\mathrm{N}^{\alpha-\beta} \ll -1. \tag{3.72}$$

Our aim in the next Section 3.3 will be to generalize to more general domains U, in particular to holed disks

$$\mathrm{U} = \mathbf{D}(0,\rho) \setminus \bigcup_{j=1}^{\mathrm{N}} \overline{\mathbf{D}(c_j,\rho_j)} \supset \mathbf{D}(0,\sigma) \tag{3.73}$$

($j = 1, \ldots, \mathrm{N}$, $\mathbf{D}(c_j,\rho_j) \subset \mathbf{D}(0,\rho)$, $\rho_j \ll \sigma$), the bound from below (3.69) and its immediate consequence inequality (3.70).

However, the spectral properties of the Laplacian (with Dirichlet boundary conditions for example) on a domain U obtained from removing disks from a simply connected domain $\Omega \in \mathbf{R}^2 \simeq \mathbf{C}$ (say the unit disk), and, in particular, the possibility of having useful estimates such as (3.71), (3.72), depend(s) on the *number* and the *sizes* of the holes of U. This fact, well known in Electrostatics under the name of *screening effect*, was mathematically studied by Rauch and Taylor in [40] (see also [41]) where they highlight two different regimes: on the one hand, if the sizes of the holes of U are (very) small compared with their number, the spectral properties of the Laplacian on U are very similar to that of Ω; on the other hand, if the holes are not so small and become dense (in some sense) in a region $\Sigma \subset \Omega$ (which can be of codimension 1 in Ω) the spectral properties of the Laplacian are similar to those of $\Omega \setminus \Sigma$. In this latter case, the holes may act like a screen that prevents the propagation of the information "$\ln f \ll -1$ on $\mathbf{D}(0,\sigma)$" to the rest of the holed domain U. In Appendix C we illustrate this phenomenon on an example.

We now give conditions under which this screening phenomenon is not effective.

3.3. *The no-screening criterion.*

Proposition **3.1.** — *Let* U *be a domain* $\mathrm{U} = \mathbf{D}(0,\rho) \setminus (\bigcup_{1\leq j\leq \mathrm{N}} \overline{\mathbf{D}(z_j,\varepsilon_j)})$, $\overline{\mathbf{D}(z_j,\varepsilon_j)} \subset \mathbf{D}(0,\rho)$ *(*$\rho \in]0,1[$*) and let* $\mathrm{B} \subset \mathrm{U}$, $\mathrm{B} = \mathbf{D}(0,\sigma)$. *Assume that* $f \in \mathcal{O}(\mathrm{U})$ *satisfies*

$$\|f\|_{\mathrm{U}} \leq 1$$

and

$$\|f\|_{\partial\mathrm{B}} \leq m.$$

Then for any point $z \in \widehat{\mathrm{U}} := \mathbf{D}(0, \rho) \setminus (\bigcup_{1 \le j \le \mathrm{N}} \mathbf{D}(z_j, d_j))$, $2\varepsilon_j < d_j < 1$

$$\ln|f(z)| \le \left(\frac{\ln(|z|/\rho)}{\ln(\sigma/\rho)} - \sum_{j=1}^{\mathrm{N}} \frac{\ln(d_j/2\rho)}{\ln(\varepsilon_j/\rho)} \right) \ln m. \tag{3.74}$$

Proof. — Replacing z/ρ by z, z_j/ρ by z_j, σ/ρ by σ, ε_j/ρ by ε_j and d_j/ρ by d_j, we can reduce to the case $\rho = 1$. We then denote $\mathrm{D} = \mathbf{D}(0, 1)$, $\mathrm{D}_j = \mathbf{D}(z_j, \varepsilon_j)$, $\mathrm{B} = \mathbf{D}(0, \sigma)$.

By Poisson-Jensen formula

$$\begin{aligned} \ln|f(z)| &\le \int_{\partial(\mathrm{U} \setminus \mathrm{B})} \ln|f(w)| d\omega_{\mathrm{U} \setminus \overline{\mathrm{B}}}(z, w) \\ &\le \omega_{\mathrm{U} \setminus \overline{\mathrm{B}}}(z, \partial \mathrm{B}) \ln m. \end{aligned} \tag{3.75}$$

We now compare $\omega_{\mathrm{U} \setminus \overline{\mathrm{B}}}(z, \partial \mathrm{B})$ with $\omega_{\mathrm{D} \setminus \overline{\mathrm{B}}}(z, \partial \mathrm{B})$. We observe that the function $z \mapsto \omega_{\mathrm{U} \setminus \overline{\mathrm{B}}}(z, \partial \mathrm{B})$ is the unique harmonic function defined on $\mathrm{U} \setminus \overline{\mathrm{B}}$ which is 1 on $\partial \mathrm{B}$ and 0 on $\partial \mathrm{D} \cup \partial(\mathrm{D} \setminus \mathrm{U})$; since $\partial(\mathrm{U} \setminus \overline{\mathrm{B}}) = \partial \mathrm{B} \cup \partial \mathrm{D} \cup \partial(\mathrm{D} \setminus \mathrm{U})$ we deduce by the Maximum Principle that it takes its values in $[0, 1]$. Similarly, the function $z \mapsto \omega_{\mathrm{D} \setminus \overline{\mathrm{B}}}(z, \partial \mathrm{B})$ is the unique harmonic function defined on $\mathrm{D} \setminus \overline{\mathrm{B}}$ which is 1 on $\partial \mathrm{B}$ and 0 on $\partial \mathrm{D}$, hence it takes also its values in $[0, 1]$. So

$$v(\cdot) := \omega_{\mathrm{U} \setminus \overline{\mathrm{B}}}(\cdot, \partial \mathrm{B}) - \omega_{\mathrm{D} \setminus \overline{\mathrm{B}}}(\cdot, \partial \mathrm{B}) \tag{3.76}$$

is a harmonic function defined on $\mathrm{U} \setminus \overline{\mathrm{B}}$, $-1 \le v \le 1$, which is 0 on $\partial \mathrm{B} \cup \partial \mathrm{D}$.

For $1 \le j \le \mathrm{N}$, let v_j be the harmonic function defined on $\mathrm{D} \setminus (\overline{\mathrm{B}} \cup \overline{\mathrm{D}_j})$ which is 0 on $\partial(\mathrm{D} \setminus \overline{\mathrm{B}}) = \partial \mathrm{D} \cup \partial \mathrm{B}$ and -1 on $\partial \mathrm{D}_j$; by the Maximum Principle $-1 \le v_j \le 0$.

Lemma **3.2.** — *The function* $\sum_{j=1}^{\mathrm{N}} v_j$ *is harmonic on* $\mathrm{U} \setminus \overline{\mathrm{B}}$ *and on this set*

$$\sum_{j=1}^{\mathrm{N}} v_j \le v.$$

Proof. — We notice that the function $\sum_{j=1}^{\mathrm{N}} v_j$ is defined and harmonic on $\mathrm{D} \setminus (\overline{\mathrm{B}} \cup \bigcup_{j=1}^{\mathrm{N}} \overline{\mathrm{D}_j}) = \mathrm{U} \setminus \overline{\mathrm{B}}$. We want to compare v and $\sum_{j=1}^{\mathrm{N}} v_j$ on the boundary $\partial(\mathrm{U} \setminus \overline{\mathrm{B}}) = \partial \mathrm{D} \cup \partial \mathrm{B} \cup \partial(\mathrm{D} \setminus \mathrm{U})$. On $\partial \mathrm{D} \cup \partial \mathrm{B}$ the two functions v and $\sum_{j=1}^{\mathrm{N}} v_j$ are equal (they are both equal to 0). To compare them on $\partial(\mathrm{D} \setminus \mathrm{U})$ we notice that $\partial(\mathrm{D} \setminus \mathrm{U}) \subset \bigcup_{j=1}^{\mathrm{N}} \partial \mathrm{D}_j$ and since $v_{j \mid \partial \mathrm{D}_j} = -1$ and for $i \neq j$, $v_i \le 0$ we have at each point $z \in \partial(\mathrm{D} \setminus \mathrm{U})$ which is in $\partial \mathrm{D}_j$, $\sum_{i=1}^{\mathrm{N}} v_i(z) \le -1$ hence $\sum_{i=1}^{\mathrm{N}} v_{i \mid \partial(\mathrm{D} \setminus \mathrm{U})} \le -1$. But we have seen that $-1 \le v \le 1$ on $\mathrm{U} \setminus \mathrm{B}$. We have thus proven that on $\partial(\mathrm{U} \setminus \mathrm{B})$ one has $\sum_{j=1}^{\mathrm{N}} v_j \le v$ and we conclude the proof by the Maximum Principle. □

Because of the Maximum Principle, one has on $D \setminus (\overline{B} \cup \overline{D}_j)$

$$-\frac{\ln|z - z_j| - \ln 2}{\ln \varepsilon_j} \le v_j(z).$$

Using Lemma 3.2 we hence get for $z \in \widehat{U}$,

$$v(z) \ge \sum_{j=1}^{N} v_j(z) \ge -\sum_{j=1}^{N} \frac{\ln|z - z_j| - \ln 2}{\ln \varepsilon_j} \ge -\sum_{j=1}^{N} \frac{\ln(d_j/2)}{\ln \varepsilon_j}.$$

On the other hand

$$\omega_{D\setminus B}(z, B) = \frac{\ln|z|}{\ln \sigma},$$

so that from (3.76) one has for $z \in \widehat{U}$

$$\omega_{U\setminus \overline{B}}(z, B) \ge \frac{\ln|z|}{\ln \sigma} - \sum_{j=1}^{N} \frac{\ln(d_j/2)}{\ln \varepsilon_j}.$$

Finally since $\ln m \le 0$, (3.75) gives that for any $z \in \widehat{U}$

$$\ln|f(z)| \le \left(\frac{\ln|z|}{\ln \sigma} - \sum_{j=1}^{N} \frac{\ln(d_j/2)}{\ln \varepsilon_j}\right) \ln m. \qquad \square$$

In particular, if for example

$$\sum_{j=1}^{N} \frac{\ln(d_j/2)}{\ln \varepsilon_j} \le (1/2)\frac{\ln|z|}{\ln \sigma}$$

then $\ln|f(z)| \le (1/2)\frac{\ln|z|}{\ln \sigma} \ln m$, an inequality which is quite similar to (3.70).

3.4. *Good triples.*

Definition **3.3.** — *Let* U, U_1, U_2 *be three nonempty open sets of* $\mathbf{C}$ *such that,*

$$U_1 \subset U, \qquad U_2 \subset U.$$

We say that the triple (U, U_1, U_2) *is* A*-good* ($A > 0$) *if for any* $f \in \mathcal{O}(U)$ *such that* $\sup_U |f| \le 1$, *one has*

$$\ln \|f\|_{U_1} \le A \ln \|f\|_{U_2}.$$

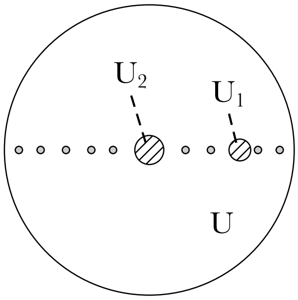

FIG. 8. — A triple (U, U_1, U_2)

Remark **3.1.** — Notice that if there exists an open set $U' \subset U$, $U_1 \subset U'$, $U_2 \subset U'$ such that (U', U_1, U_2) is A-good, then (U, U_1, U_2) is also A-good.

Remark **3.2.** — In general the fact (U, U_1, U_2) is A-good does not imply that (U, U_2, U_1) is A′-good with A and A′ comparable. For example, if $U = \mathbf{D}(0, 1)$, $U_1 = \mathbf{D}(1/4, \sigma_1)$ $U_2 = \mathbf{D}(3/4, \sigma_2)$ with $\sigma_1, \sigma_2 < 1/10$, (U, U_1, U_2) is $C/|\ln \sigma_2|$-good while (U, U_2, U_1) is $C/|\ln \sigma_1|$-good.

We denote by $\mathbf{A}(z; \lambda_1, \lambda_2)$, $0 < \lambda_1 < \lambda_2$, the annulus $\mathbf{D}(z, \lambda_2) \setminus \overline{\mathbf{D}}(z, \lambda_1)$.

Here is an immediate corollary of Proposition 3.1:

Corollary **3.4.** — *Assume that the assumptions of Proposition 3.1 hold with* $\sigma = \rho^b/2$ *(*$b > 1$*). Then for all* $1 \le i \le N$ *such that* $d_i > 20\varepsilon_i$ *and* $\mathbf{D}(z_i, d_i) \subset \mathbf{D}(0, e^{-\delta}\rho)$ *(*$\delta > 0$*), the triple*

$$\Big(U, \mathbf{A}(z_i; (d_i/10), d_i), \mathbf{D}(0, \rho^b/2)\Big)$$

is A*-good with*

$$A = \frac{\delta}{b|\ln \rho|} - \sum_{j=1}^{N} \frac{\ln(d_j/20\rho)}{\ln(\varepsilon_j/\rho)}.$$

4. Symplectic diffeomorphisms on holed domains

4.1. *Cartesian Coordinates (CC) and Action-Angle variables (AA).* — We define on $\mathbf{R}^2 := \{(x, y),\ x, y \in \mathbf{R}\}$ (resp. $\mathbf{T} \times \mathbf{R} := \{(\theta, r),\ \theta \in \mathbf{T},\ r \in \mathbf{R}\}$) the canonical symplectic structure (area) $\beta_{\mathbf{R}}^{CC*} := dx \wedge dy$ (resp. $\beta_{\mathbf{R}}^{AA} := d\theta \wedge dr$). This space as well as its symplectic structure can be complexified: the space $\mathbf{C}^2 := \{(x, y),\ x,\ y \in \mathbf{C}\}$ (resp. $\mathbf{T}_\infty \times \mathbf{C} := \{(\theta, r),\ \theta \in \mathbf{T}_\infty,\ r \in \mathbf{C}\}$) carries the symplectic structure $\beta_{\mathbf{C}}^{CC*} := dx \wedge dy$ (resp.

$\beta_{\mathbf{C}}^{\rm AA} := d\theta \wedge dr$) and the involution σ_1 (resp. σ_3) defined in (2.42) preserves $(\mathbf{C}^2, \beta_{\mathbf{C}}^{\rm CC*})$ (resp. $(\mathbf{T}_\infty \times \mathbf{C}, \beta_{\mathbf{C}}^{\rm AA})$) and fixes $(\mathbf{R}^2, \beta_{\mathbf{R}}^{\rm CC*})$ (resp. $(\mathbf{T} \times \mathbf{R}, \beta_{\mathbf{R}}^{\rm AA})$).

When working in the elliptic fixed point case, it will be more convenient to use other Cartesian coordinates. Let's introduce the (holomorphic) complex change of coordinates $\varphi : \mathbf{C}^2 \to \mathbf{C}^2$, $\varphi : (x, y) \mapsto (z, w)$,

$$\textbf{(4.77)} \qquad \begin{cases} z = \frac{1}{\sqrt{2}}(x + iy) \\ w = \frac{i}{\sqrt{2}}(x - iy) \end{cases} \Longleftrightarrow \begin{cases} x = \frac{1}{\sqrt{2}}(z - iw) \\ y = \frac{-i}{\sqrt{2}}(z + iw). \end{cases}$$

We see that (σ_2 is as in (2.42)) with the notations of Section 2.1

$$dx \wedge dy = \varphi^*(dz \wedge dw), \qquad \varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_2, \qquad \varphi(\mathrm{W}_{h,\mathrm{U}}^{\rm CC*}) = \mathrm{W}_{h,\mathrm{U}}^{\rm CC}.$$

We shall denote $(\mathrm{M}^{\rm CC}, \beta^{\rm CC}, \sigma_2)$, *resp.* $(\mathrm{M}^{\rm CC*}, \beta^{\rm CC^*}, \sigma_1)$, (CC stands for Cartesian Coordinates) the space $\mathbf{C}^2$ endowed with the symplectic structure $\beta^{\rm CC} := dz \wedge dw$, *resp.* $\beta^{\rm CC*} = dx \wedge dy$, and the involution σ_2, *resp.* σ_1. Similarly, $(\mathrm{M}^{\rm AA}, \beta^{\rm AA}, \sigma_3)$ (AA for Action-Angle coordinates) is the space $\mathbf{T}_\infty \times \mathbf{C}$ endowed with the symplectic structure $\beta^{\rm AA} := d\theta \wedge dr$ and the involution σ_3. We shall use for short the generic notation $(\mathrm{M}, \beta, \sigma)$ to denote either of the preceding sets endowed with their symplectic structure and involution. We also use the notation $\mathrm{M}_{\mathbf{R}}$ or $(\mathrm{M})_{\mathbf{R}}$ for $\mathrm{M} \cap \sigma(\mathrm{M})$. The 2-form β restricted to $\mathrm{M}_{\mathbf{R}}$ is still a symplectic form. We shall call the *origin* O in $\mathrm{M}_{\mathbf{R}}$, the set $\mathrm{O} = \{(0, 0)\}$ if $\mathrm{M} = \mathbf{C}^2 = \mathrm{M}^{\rm CC}$ or $\mathrm{M}^{\rm CC^*}$ and $\mathrm{O} = \mathbf{T} \times \{0\}$ if $\mathrm{M} = \mathrm{M}^{\rm AA} = \mathbf{T}_\infty \times \mathbf{C}$.

If W is a nonempty open set of M (resp. $\mathrm{M}_{\mathbf{R}}$) we say that $f \in \mathrm{Diff}^{\mathcal{O}}(\mathrm{W})$ (resp. $f \in \mathrm{Diff}^{\mathrm{C}^1}(\mathrm{W})$) is *symplectic* if it preserves the canonical symplectic form β: $f^*\beta = \beta$. We denote by $\mathrm{Symp}^{\mathcal{O}}(\mathrm{W})$ (resp. $\mathrm{Symp}^{\mathrm{C}^1}(\mathrm{W})$) the set of such symplectic holomorphic (resp. C^1) diffeomorphisms. If furthermore $f \circ \sigma = \sigma \circ f$ we write $f \in \mathrm{Symp}^{\mathcal{O}}_\sigma(\mathrm{W})$. We shall say that a symplectic diffeomorphism f is *exact symplectic* if there exists a 1-form λ, the *Liouville form*, such that $d\lambda = \beta$ and $f^*\lambda - \lambda$ is exact: there exists a function S such that $f^*\lambda - \lambda = d\mathrm{S}$; S is called the *generating function* of f (w.r.t. λ). We then denote $f \in \mathrm{Symp}^{\mathcal{O}}_{ex.,\sigma}(\mathrm{W})$ (resp. $f \in \mathrm{Symp}^{\mathrm{C}^1}_{ex.}(\mathrm{W})$). In our case the relevant Liouville forms will be

$$\textbf{(4.78)} \qquad \text{(AA)}\ \lambda = r d\theta, \quad \text{(CC)}\ \lambda = (1/2)(w dz - z dw), \quad \text{(CC*)}\ \lambda = (1/2)(x dy - y dx).$$

Let $\mathrm{W} \subset \mathrm{M}$ be σ-symmetric ($\sigma(\mathrm{W}) = \mathrm{W}$) and such that $(\mathrm{W})_{\mathbf{R}} := \mathrm{W} \cap \sigma(\mathrm{W}) = \mathrm{W} \cap \mathrm{M}_{\mathbf{R}}$ is a nonempty open set of $\mathrm{M}_{\mathbf{R}}$. Then, if $f \in \mathrm{Symp}^{\mathcal{O}}_{ex.,\sigma}(\mathrm{W})$, its restriction $f_{|(\mathrm{W})_{\mathbf{R}}} : \mathrm{M}_{\mathbf{R}} \supset (\mathrm{W})_{\mathbf{R}} \to f((\mathrm{W})_{\mathbf{R}}) \subset \mathrm{M}_{\mathbf{R}}$ defines a real analytic (exact) symplectic diffeomorphism. If $\mathrm{S} \subset \mathrm{W}$ is f-invariant ($f(\mathrm{S}) = \mathrm{S}$) the set $(\mathrm{S})_{\mathbf{R}} := \mathrm{S} \cap \mathrm{M}_{\mathbf{R}}$ is also left invariant by $f_{|(\mathrm{W})_{\mathbf{R}}}$. Notice that if $\mathrm{U} \subset \mathbf{C}$ is a real symmetric open set such that $\mathrm{U} \cap \mathbf{R} \neq \emptyset$ we have

$$\begin{cases} (\mathrm{W}_{h,\mathrm{U}}^{\rm AA})_{\mathbf{R}} = (\mathbf{T}_h \times \mathrm{U})_{\mathbf{R}} = \mathbf{T} \times (\mathrm{U} \cap \mathbf{R}) = \mathrm{W}_{0,\mathrm{U}\cap\mathbf{R}}^{\rm AA} \\ (\mathrm{W}_{h,\mathrm{U}}^{\rm CC})_{\mathbf{R}} = \{(z, w) \in \mathrm{W}_{0,\mathrm{U}\cap\mathbf{R}_+}^{\rm CC},\ w = i\bar{z}\} = (\mathrm{W}_{0,\mathrm{U}\cap\mathbf{R}_+}^{\rm CC})_{\mathbf{R}} \\ (\mathrm{W}_{h,\mathrm{U}}^{\rm CC*})_{\mathbf{R}} = \{(x, y) \in \mathbf{R}^2,\ \frac{x^2+y^2}{2} \in \mathrm{U} \cap \mathbf{R}_+\} = \mathrm{W}_{0,\mathrm{U}\cap\mathbf{R}_+}^{\rm CC*}. \end{cases}$$

In any case $(W_{h,U})_{\mathbf{R}} = \{r \in U\} \cap M_{\mathbf{R}} = \{r \in U \cap \mathbf{R}\} \cap M_{\mathbf{R}}$.

There are symplectic changes of coordinates $\psi_\pm$ that allow to pass from the (z, w)-coordinates ((CC)-coordinates) to the (θ, r)-coordinates ((AA)-coordinates). They are defined as follows. The maps $r \mapsto r^{1/2}$, $te^{is} \mapsto t^{1/2}e^{is/2}$ for $t > 0$ and $-\pi < s < \pi$ (*resp.* for $t > 0$ and $0 < s < 2\pi$) define holomorphic functions on $\mathbf{C} \setminus \mathbf{R}_-$ (*resp.* on $\mathbf{C} \setminus \mathbf{R}_+$). We can thus define the biholomorphic diffeomorphisms

$$\mathbf{T}_\infty \times (\mathbf{C} \setminus \mathbf{R}_\pm) \ni (\theta, r) \xrightarrow{\psi_\pm} (z, w) \in \{(z, w) \in \mathbf{C}^2,\ -izw \notin \mathbf{R}_\pm\}$$

$$(\mathbf{4.79}) \qquad \begin{cases} z = e^{i\pi/4} r^{1/2} e^{-i\theta} \\ w = e^{i\pi/4} r^{1/2} e^{i\theta} \end{cases} \Longleftrightarrow \begin{cases} r = -izw \\ e^{i\theta} = e^{-i\pi/4} \frac{w}{(-izw)^{1/2}} = e^{i\pi/4} \frac{(-izw)^{1/2}}{z} \end{cases}$$

which satisfy

$$dz \wedge dw = d\theta \wedge dr \text{ and } \psi_\pm \circ \sigma_2 \circ \psi_\pm^{-1} = \sigma_3.$$

Notice that if $h > 0$

$$(\mathbf{4.80}) \qquad \mathbf{T}_h \times (\mathbf{D}(0, \rho) \setminus \mathbf{R}_\pm) \xrightarrow{\psi_\pm} \left\{ (z, w) \in \mathbf{C}^2, \begin{cases} -izw \in \mathbf{D}(0, \rho) \setminus \mathbf{R}_\pm \\ e^{-2h} < |z/w| < e^{2h} \end{cases} \right\}$$

hence with the notations of Section 2.1

$$(\mathbf{4.81}) \qquad W^{CC}_{h, U \setminus \mathbf{R}_\pm} \supset \psi_\pm(W^{AA}_{h, U \setminus \mathbf{R}_\pm}).$$

4.2. *Symplectic vector fields.* — If $(M, \beta) = (M^{CC}, \beta)$ or (M^{AA}, β) and $F \in \mathcal{O}_\sigma(M)$ we define the holomorphic *symplectic* vector field X by $i_X \beta = dF$. If J is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ one has

$$X_F = J\nabla F.$$

We denote by $\phi^t_{J\nabla F}$ the flow at time $t \in \mathbf{R}$ of the vector field $J\nabla F$ and $\Phi_F = \phi^1_{J\nabla F}$ its time 1-map. It is a symplectic diffeomorphism.

If $G : M \to \mathbf{R}$ or $\mathbf{C}$ is another smooth observable we define the *Poisson bracket* of F and G by the formula $\{F, G\} = \beta(X_F, X_G)$ or equivalently

$$\{F, G\} := \langle \nabla F, J\nabla G \rangle.$$

One then has

$$\frac{d}{dt}(G \circ \Phi^t_F)_{|t=0} = L_{J\nabla F} G = \{F, G\}, \qquad [L_{X_F}, L_{X_G}] = L_{X_{\{F,G\}}}.$$

If f is a symplectic diffeomorphism one has

$$(\mathbf{4.82}) \qquad f \circ \Phi_F \circ f^{-1} = \Phi_{f_* F}, \qquad \text{where } f_* F = (f^{-1})^* F = F \circ f^{-1}.$$

4.3. *Integrable models.* — We assume that $(\mathrm{M}, \beta, \sigma)$ is $(\mathbf{C}^2, dx\wedge dy, \sigma_1)$, $(\mathbf{C}^2, dz\wedge dw, \sigma_2)$ or $(\mathbf{T}_\infty\times\mathbf{C}, d\theta\wedge dr, \sigma_3)$. In all these examples there exists a natural (Lagrangian) foliation given by the level lines of the observable $r:\mathrm{M}\to\mathbf{C}$

$$r(x,y)=\frac{x^2+y^2}{2},\quad r(z,w)=-izw,\quad r(\theta,r)=r, \tag{4.83}$$

which has the property that for every $m\in\mathrm{M}$, such that $r(m)\in\mathbf{R}$, the map $\mathbf{R}\ni t\mapsto \phi^t_{\mathrm{J}\nabla r}(m)$ is 2π-periodic. In particular, for $c\in\mathbf{R}$, the set $\{r=c\}\subset\mathrm{M}$ is itself foliated by the 2π-periodic orbits of the flow $\phi^t_{\mathrm{J}\nabla r}$; they are either points or homeomorphic to $\mathbf{S}^1$. We shall say that a symplectic diffeomorphism of M is *integrable* if it is symplectically conjugated to a diffeomorphism that leaves globally invariant each level line of the preceding function r. It is not difficult to see that a diffeomorphism satisfying the previous condition is of the form Φ_{H} where $\mathrm{H}=\Omega\circ r$.

Let U be a σ-symmetric holed domain of $\mathbf{C}$ and $\Omega\in\mathcal{O}_\sigma(\mathrm{U})$. Then,

$$\begin{cases}(\mathrm{CC}):\ \Phi_\Omega(z,w)=(e^{-i\partial\Omega(r)}z, e^{i\partial\Omega(r)}w),\\ (\mathrm{AA}):\ \Phi_\Omega(\theta,r)=(\theta+\partial\Omega(r), r),\\ (\mathrm{CC}*):\ \Phi_\Omega(x,y)=(\Re(e^{-i\partial\Omega(r)}(x+iy)), \Im(e^{-i\partial\Omega(r)}(x+iy)))\end{cases} \tag{4.84}$$

and in any case

$$\Phi_\Omega(\mathrm{W}_{\widetilde{h},\mathrm{U}})\subset\mathrm{W}_{h,\mathrm{U}},\qquad \widetilde{h}=h-\|\Im(\partial\Omega)\|_{\mathrm{U}}.$$

On the other hand, since Ω is σ-symmetric, one has whenever U is σ-symmetric,

$$\Phi_\Omega((\mathrm{W}_{h,\mathrm{U}})_{\mathbf{R}})=((\mathrm{W}_{h,\mathrm{U}})_{\mathbf{R}}).$$

Notice that in all cases Φ_Ω is an integrable diffeomorphism of M.

4.4. *KAM circles.* — A *circle* of $\mathrm{M}_{\mathbf{R}}$ ($\mathrm{M}_{\mathbf{R}}$ equals $\mathrm{M}^{\mathrm{CC}*}_{\mathbf{R}}=\mathbf{R}^2$, $\mathrm{M}^{\mathrm{CC}}_{\mathbf{R}}$, $\mathrm{M}^{\mathrm{AA}}_{\mathbf{R}}=\mathbf{T}\times\mathbf{R}$) is any set of the form $(\{r=c\})_{\mathbf{R}}=(\{r=c)\}\cap\mathrm{M}_{\mathbf{R}}$, $c\in\mathbf{R}$, of cardinal >1 (r is the observable of (4.83)). In the (AA) *resp.* (CC*) cases this set coincides with the usual circle $\mathbf{T}\times\{r=c\}$ *resp.* $\{(x,y)\in\mathbf{R}^2,\ (1/2)(x^2+y^2)=r\}$; in the (CC) or (CC*) cases $(\{r=c\})_{\mathbf{R}}$ is a circle if and only if $c>0$ (it is empty if $c<0$ and reduced to $\{(0,0\}$ if $c=0$).

Let W be an open subset of $\mathrm{M}_{\mathbf{R}}$ and $f\in\mathrm{Symp}^{\mathrm{C}^1}_{ex.}(\mathrm{W})$ a C^1 symplectic diffeomorphism $\mathrm{W}\mapsto f(\mathrm{W})$. For example f could be the restriction on $\mathrm{W}=(\mathrm{W}_{h,\mathrm{U}})_{\mathbf{R}}$ of $f\in\mathrm{Symp}^{\mathcal{O}}_{ex.,\sigma}(\mathrm{W}_{h,\mathrm{U}})$, $\mathrm{W}_{h,\mathrm{U}}\subset\mathrm{M}$. A *KAM-circle* (or *KAM-curve*) for f is the image $g((\{r=c\})_{\mathbf{R}})\subset\mathrm{W}$ of a circle $(\{r=c\})_{\mathbf{R}}$, $c\in\mathbf{R}$, by a C^1 symplectic diffeomorphism $g:\mathrm{M}_{\mathbf{R}}\to\mathrm{M}_{\mathbf{R}}$ fixing the origin ($g(\{r=0\}_{\mathbf{R}})=\{r=0\}_{\mathbf{R}}$) and such that

$$g^{-1}\circ f\circ g=\Phi_{2\pi\omega r}+\mathrm{O}(r-c),\qquad \omega\in\mathbf{R}\setminus\mathbf{Q}.$$

The set $g((\{r=c\})_{\mathbf{R}})\subset \mathrm{W}$ is then f-invariant, homeomorphic to $\mathbf{S}^1$ and non homotopically trivial in the following sense: in the (AA)-case it is homotopic to $\{r=0\}_{\mathbf{R}}=\mathbf{T}\times\{0\}$ and in the (CC) or (CC*) case it has degree ± 1 w.r.t. to the origin $\{r=0\}_{\mathbf{R}}=\{(0,0)\}$. Moreover, the restriction of f on $g((\{r=c\})_{\mathbf{R}})\subset \mathrm{W}$ is conjugated to a rotation on a circle with frequency $\omega\in\mathbf{R}$.

Notation **4.1.** — *We denote by* $\mathcal{G}(f,\mathrm{W})$ *the set of* f*-invariant KAM-circles* $\gamma\subset(\mathrm{W})_{\mathbf{R}}$ *and by* $\mathcal{L}(f,\mathrm{W})\subset(\mathrm{W})_{\mathbf{R}}$ *their union:* $\mathcal{L}(f,\mathrm{W})=\bigcup_{\gamma\in\mathcal{G}(f,\mathrm{W})}\gamma$.

Remark **4.1.** — Let $g,f,f_1,f_2:\mathrm{M}_{\mathbf{R}}\to\mathrm{M}_{\mathbf{R}}$ be C^1 symplectic diffeomorphisms where $g(\{r=0\}_{\mathbf{R}})=\{r=0\}_{\mathbf{R}}$. Then,

(1) If $\mathrm{A}\subset\mathrm{B}\subset(\mathrm{M})_{\mathbf{R}}$, then $\mathcal{L}(f,\mathrm{A})\subset\mathcal{L}(f,\mathrm{B})$.
(2) If f_1,f_2 coincide on a set A, $\mathcal{L}(f_1,\mathrm{A})=\mathcal{L}(f_2,\mathrm{A})$.
(3) For any set $\mathrm{A}\subset\mathrm{M}_{\mathbf{R}}$

$$g(\mathcal{L}(f,\mathrm{A}))=\mathcal{L}(g\circ f\circ g^{-1},g(\mathrm{A})). \tag{4.85}$$

(4) If $g^{-1}\circ f_1\circ g$ and f_2 coincide on a set A one has

$$\mathcal{L}(f_1,g(\mathrm{A}))=g(\mathcal{L}(f_2,\mathrm{A})). \tag{4.86}$$

Definition **4.2.** — *If* $\mathrm{A}\subset\mathbf{C}$ *we define* $\mathrm{W}_{\mathrm{A}}=\{r\in\mathrm{A}\}\cap\mathrm{M}_{\mathbf{R}}=\{r\in\mathrm{A}\cap\mathbf{R}\}\cap\mathrm{M}_{\mathbf{R}}$.

Let us now state a criterion that ensures the existence of KAM-circles. Assume that there exist

$$\emptyset\neq\mathrm{L}\subset\mathrm{A}=\mathrm{I}\setminus\bigcup_{j\in\mathrm{J}}\mathrm{I}_j\subset\widetilde{\mathrm{A}}\subset\mathbf{R},$$

where L is compact and A is of the form $\mathrm{I}\setminus\bigcup_{j\in\mathrm{J}}\mathrm{I}_j$ where $\mathrm{I}\subset\mathbf{R}$ is an interval and the I_j are pairwise disjoint intervals.

Proposition **4.3.** — *Let* $f\in\mathrm{Symp}^{\mathrm{C}^1}(\mathrm{W}_{\widetilde{\mathrm{A}}})$ *and suppose that there exist* $\Omega\in\mathrm{C}^1(\mathbf{R})$ *and a* C^1 *symplectic diffeomorphism* $g:\mathrm{M}_{\mathbf{R}}\to\mathrm{M}_{\mathbf{R}}$ *fixing the origin,* $\|g-id\|_{\mathrm{C}^1}\leq\mathrm{C}^{-1}$ *(*C *depends only on* M*), such that*

$$\textit{on } \mathrm{W}_{\mathrm{L}}\qquad g^{-1}\circ f\circ g=\Phi_{\Omega(r)}\qquad \textit{and}\qquad g(\mathrm{W}_{\mathrm{L}})\subset\mathrm{W}_{\widetilde{\mathrm{A}}}.$$

Then, if $\sum_{j\in\mathrm{J}}|\mathrm{I}_j|^{1/2}\leq 1$, *one has*

$$\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}(\mathrm{W}_{\mathrm{A}}\setminus\mathcal{L}(f,\mathrm{W}_{\widetilde{\mathrm{A}}}))\leq\mathrm{C}\times(\mathrm{Leb}_{\mathbf{R}}(\mathrm{A}\setminus\mathrm{L})+\|g-id\|_{\mathrm{C}^0}^{1/2}).$$

Proof. — Since $\mathrm{W}_{\mathrm{L}}=\mathcal{L}(\Phi_{\Omega(r)},\mathrm{W}_{\mathrm{L}})$ one has from (4.86) $g(\mathrm{W}_{\mathrm{L}})=g(\mathcal{L}(\Phi_{\Omega(r)},\mathrm{W}_{\mathrm{L}})=\mathcal{L}(f,g(\mathrm{W}_{\mathrm{L}}))$ and since $g(\mathrm{W}_{\mathrm{L}})\subset\mathrm{W}_{\widetilde{\mathrm{A}}}$ one has $g(\mathrm{W}_{\mathrm{L}})\subset\mathcal{L}(f,\mathrm{W}_{\widetilde{\mathrm{A}}})$. On the other hand if we define E by $W_A = W_L \cup E$, one has $g(W_A) = g(W_L) \cup g(E)$ and thus $g(W_A) \subset \mathcal{L}(f, W_{\tilde{A}}) \cup g(E)$. We therefore have

$$\mathrm{Leb}_{M_{\mathbf{R}}}(W_A \smallsetminus \mathcal{L}(f, W_{\tilde{A}})) \lesssim \mathrm{Leb}_{M_{\mathbf{R}}}(g(E)) + \mathrm{Leb}_{M_{\mathbf{R}}}(W_A \triangle g(W_A)).$$

Since $A = I \smallsetminus \bigcup_{j\in J} I_j$ and $\sum_{j\in J} |I_j|^{1/2} \leq 1$, Lemma J.1 from the Appendix yields $\mathrm{Leb}_{M_{\mathbf{R}}}(W_A \triangle g(W_A)) \leq \|g - id\|_{C^0}^{1/2}$ and since $\mathrm{Leb}_{M_{\mathbf{R}}}(g(E)) = \mathrm{Leb}_{M_{\mathbf{R}}}(E)$ we get the conclusion. □

4.5. *Generating functions.* — Let $h > 0$, $U \subset \mathbf{C}$ be a real symmetric holed domain and $W^{AA}_{h,U}$ and $W^{CC}_{h,U}$ the domains defined in (2.47) and (2.46)

$$\begin{cases} W^{AA}_{h,U} = \mathbf{T}_h \times U \\ W^{CC}_{h,U} = \{(z, w) \in \mathbf{D}(0, e^h \rho_U^{1/2}) \times \mathbf{D}(0, e^h \rho_U^{1/2}),\ r := -izw \in U\}. \end{cases}$$

We shall associate to each $F \in \mathcal{O}_\sigma(W_{h,U})$ small enough a real symmetric holomorphic symplectic diffeomorphism f_F of $W_{h,U}$ which is *exact* with respect to the respective Liouville forms as defined in (4.78)). It is defined as follows: in the (AA)-case

$$f_F(\theta, r) = (\varphi, R) \iff \begin{cases} \varphi = \theta + \partial_R F(\theta, R) \\ r = R + \partial_\theta F(\theta, R) \end{cases} \tag{4.87}$$

and in the (CC)-case

$$f_F(z, w) = (\tilde{z}, \tilde{w}) \iff \begin{cases} \tilde{z} = z + \partial_{\tilde{w}} F(z, \tilde{w}) \\ w = \tilde{w} + \partial_{\tilde{w}} F(z, \tilde{w}). \end{cases} \tag{4.88}$$

Lemma **4.4.** — *There exists a constant* $\overline{C}$ *such that if* $F \in \mathcal{O}_\sigma(W_{h,U})$ *and* $0 < \delta < h$ *satisfy*

$$\overline{C}(\delta\, \underline{d}(W_{h,U}))^{-2} \|F\|_{W_{h,U}} < 1, \tag{4.89}$$

the map f_F *defined by (4.87), (4.88) is a real symmetric holomorphic exact symplectic diffeomorphism from* $e^{-\delta} W_{h,U}$ *onto its image and*

$$e^{-2\delta} W_{h,U} \subset f_F(e^{-\delta} W_{h,U}) \subset W_{h,U}. \tag{4.90}$$

We shall call f_F *the generating map of* F. *Moreover*

$$f_F^{-1} = f_{-F + O(\|D^2F\| \|DF\|)}. \tag{4.91}$$

Proof. — See Appendix A.1. □

Remark **4.2.** — The symplectic change of coordinates $\psi_\pm$ introduced in Section 4.1 preserves exact symplecticity: if $f^{\rm CC}$ is exact symplectic the same is true for $f^{\rm AA} = \psi_\pm^{-1} \circ f^{\rm CC} \circ \psi_\pm$. Indeed, if $\psi_\pm(\theta, r) = (z, w)$, $z = e^{i\pi/4} r^{1/2} e^{-i\theta}$, $w = e^{i\pi/4} r^{1/2} e^{i\theta}$, one computes the Liouville form $(1/2)(wdz - zdw) = rd\theta$.

Conversely, if a diffeomorphism $(\theta, r) \mapsto (\varphi, {\rm R})$ is exact symplectic and close enough to the identity, it admits this type of parametrization.

More precisely:

Lemma **4.5.** — *Let* $f \in {\rm Symp}^{\mathcal{O}}_{ex.\sigma}({\rm W}_{h,{\rm U}})$ *be an exact symplectic diffeomorphism close enough to the identity. Then, if* $\delta = \mathfrak{d}(f - id, {\rm W}_{h,{\rm U}})$ *(recall the notation (2.64)) there exists* ${\rm F} \in \mathcal{O}_\sigma(e^{-\delta}{\rm W}_{h,{\rm U}})$ *such that on* $e^{-\delta}{\rm W}_{h,{\rm U}}$ *one has*

$$f = f_{\rm F}, \qquad {\rm F} = {\rm O}(\|{\rm D}f - id\|) = \mathfrak{O}_1(f - id).$$

This F *is unique up to the addition of a constant.*

Conversely, given ${\rm F} \in \mathcal{O}({\rm W}_{h,{\rm U}})$ *one has*

$$f_{\rm F} = \Phi_{\rm F} \circ f_{\mathfrak{O}_2({\rm F})} = id + {\rm J}\nabla {\rm F} + {\rm O}(\|{\rm D}^2{\rm F}\|\|{\rm DF}\|). \tag{4.92}$$

Proof. — See Appendix A.2. □

The composition of two exact symplectic maps is again exact symplectic and more precisely

Lemma **4.6.** — *Let* ${\rm F}, {\rm G} \in \mathcal{O}({\rm W}_{h,{\rm U}})$. *If* $\delta = \mathfrak{d}({\rm F}, {\rm G}; {\rm W}_{h,{\rm U}})$ *then on* $e^{-\delta}{\rm W}_{h,{\rm U}}$,

$$f_{\rm F} \circ f_{\rm G} = f_{{\rm F}+{\rm G}+{\rm O}(\|{\rm DF}\|_{h,{\rm U}}\|{\rm DG}\|_{h,{\rm U}})} \tag{4.93}$$

$$f_{{\rm F}+{\rm G}} = f_{{\rm F}+\|{\rm DF}\|_{h,{\rm U}}\mathfrak{O}_1({\rm G})} \circ f_{\rm G} = f_{\rm F} \circ f_{{\rm G}+\|{\rm DG}\|_{h,{\rm U}}\mathfrak{O}_1({\rm F})}. \tag{4.94}$$

In the Action-Angle case, if Ω *depends only on the variable* r *then* $\Phi_\Omega = f_\Omega$ *and*

$$\Phi_\Omega \circ f_{\rm F} = f_{\Omega+{\rm F}} \tag{4.95}$$

Proof. — See the Appendix, Section A.3. □

4.6. *Parametrization.* — We shall parametrize *perturbations* of integrable symplectic diffeomorphisms defined on a domain ${\rm W}_{h,{\rm U}}$ by

$$f = \Phi_{\Omega(r)} \circ f_{\rm F}$$

where $\Omega \in \mathcal{O}_\sigma({\rm U})$ and ${\rm F} \in \mathcal{O}_\sigma({\rm W}_{h,{\rm U}})$. Note that if $f_{\rm F} = id + {\rm O}^2(z, w)$ or $f(\theta, r) = id + ({\rm O}(r), {\rm O}(r^2))$ then:

$$\text{Case (CC)} \quad {\rm F}(z, w) = {\rm O}^3(z, w), \qquad \text{Case (AA)} \quad {\rm F}(\theta, r) = {\rm O}(r^2).$$

4.7. *Transformation by conjugation.* — We now define

$$(\mathbf{4.96}) \qquad [\Omega]\cdot Y = Y\circ\Phi_\Omega - Y.$$

Note that

$$(\mathbf{4.97}) \qquad \begin{cases} \text{(AA)-case if } Y = Y(\theta,r), \quad ([\Phi]\cdot Y)(\theta,r) = Y(\theta+\partial\Omega(r),r) - Y(\theta,r);\\ \text{(CC)-case if } Y = Y(z,w),\\ \quad ([\Phi]\cdot Y)(z,w) = Y(e^{-i\partial\Omega(r)}z, e^{i\partial\Omega(r)}w) - Y(z,w).\end{cases}$$

If $W = W_{h,U}$ is a holed domain and $\delta > 0$ we introduce the notation

$$W^{\Omega}_{h,U} = W^{\Phi_\Omega}_{h,U} := W_{h,U}\cup\Phi_\Omega(W_{h,U}).$$

The main result of this section is the following:

Proposition **4.7.** — *Let* $\Omega\in\mathcal{O}_\sigma(U)$, $F\in\mathcal{O}_\sigma(W_{h,U})$, $Y\in\mathcal{O}_\sigma(W^{\Omega}_{h,U})$. *Then, if* $\delta = \mathfrak{d}(F, W_{h,U})\cap\mathfrak{d}(Y, W^{\Omega}_{h,U})$ *there exists* $\widetilde{F}\in\mathcal{O}_\sigma(e^{-\delta}W_{h,U})$ *such that*

$$[e^{-\delta}W_{h,U}] \qquad f_Y\circ(\Phi_\Omega\circ f_F)\circ f_Y^{-1} = \Phi_\Omega\circ f_{\widetilde{F}}$$

(see the notation (2.44)) and

$$\begin{aligned}\widetilde{F} &= F + [\Omega]\cdot Y + \|DF\|_W\mathfrak{O}_1(Y)\\ &= F + [\Omega]\cdot Y + \dot{\mathfrak{O}}_2(Y,F).\end{aligned}$$

Proof. — See the Appendix, Section A.4. □

Remark **4.3.** — A direct computation shows that if $\Omega(r) = 2\pi\omega_0 r + O(r^2)$ and

$$\text{Case (CC)} \quad F(z,w) = O^k(z,w) \quad \text{and} \quad Y(z,w) = O^k(z,w),$$
$$\text{Case (AA)} \quad F(\theta,r) = O(r^k) \quad \text{and} \quad Y(\theta,r) = O(r^k)$$

then

$$\text{Case (CC)} \quad \widetilde{F}(z,w) = O^{2k-2}(z,w), \qquad \text{Case (AA)} \quad \widetilde{F}(\theta,r) = O^{2k-1}(r).$$

4.8. *Symplectic Whitney extensions.* — Let $U\subset\mathbf{C}$ be a real symmetric holed domain $W_{h,U}\subset M$, $F\in\mathcal{O}_\sigma(W_{h,U})$ and $F^{W_h}: M\to\mathbf{C}$ be a σ-symmetric C^2 Whitney extension of $(F, W_{h,U})$ (*cf.* Section 2.4). There exists a constant $C>0$ (depending only on M) such that if $\|F^{W_h}\|_{C^2(M)} < C^{-1}$, Equations (4.87), (4.88) define a C^1-diffeomorphism $f_{F^{W_h}}: M\to M$ such that

$$(\mathbf{4.98}) \qquad \max(\|f_{F^{W_h}} - id\|_{C^1(M)}, \|f^{-1}_{F^{W_h}} - id\|_{C^1(M)}) \le C^{-1}\|F^{W_h}\|_{C^2(M)}.$$

Note that $f_{\mathrm{F}^{\mathrm{W}_h}}$ and $f_{\mathrm{F}^{\mathrm{W}_h}}^{-1}$ are C^1 σ-symmetric extensions of $(f_{\mathrm{F}}, e^{-\delta}\mathrm{W}_{h,\mathrm{U}})$ and $(f_{\mathrm{F}}^{-1}, e^{-\delta}\mathrm{W}_{h,\mathrm{U}})$ for any δ satisfying (4.89), *cf.* Lemma 4.4.

In general, the diffeomorphism $f_{\mathrm{F}^{\mathrm{W}_h}}$ is *not symplectic* on M but since $\mathrm{F}^{\mathrm{W}_h}$ takes real values on $\mathrm{M}_{\mathbf{R}}$, $f_{\mathrm{F}^{\mathrm{W}_h}} : \mathrm{M}_{\mathbf{R}} \to \mathrm{M}_{\mathbf{R}}$ is an exact symplectic diffeomorphism of $\mathrm{M}_{\mathbf{R}}$.

Notation **4.8.** — *We shall denote by* $\widetilde{\mathrm{Symp}}_\sigma(\mathrm{W}_{h,\mathrm{U}})$, *resp.* $\widetilde{\mathrm{Symp}}_{ex.,\sigma}(\mathrm{W}_{h,\mathrm{U}})$, *the set of* C^1 σ*-symmetric diffeomorphisms* $\mathrm{M} \to \mathrm{M}$ *that are in* $\mathrm{Symp}_\sigma^{\mathcal{O}}(\mathrm{W}_{h,\mathrm{U}})$, *resp.* $\mathrm{Symp}_{ex,s}^{\mathcal{O}}(\mathrm{W}_{h,\mathrm{U}})$, *(hence holomorphic on* $\mathrm{W}_{h,\mathrm{U}}$*) and symplectic, resp. exact symplectic, when restricted to* $\mathrm{M}_{\mathbf{R}} \to \mathrm{M}_{\mathbf{R}}$.

5. Cohomological equations and conjugations

Our aim in this section is to provide a unified treatment, both in the (AA) and (CC) cases, of the resolution of the (co)homological equations (Proposition 5.3) involved in the Fundamental conjugation step (Proposition 5.5) that we shall use to construct all our different Normal Forms (for instance the approximate Birkhoff Normal Form of Section 6, the KAM Normal Forms of Section 7 and the resonant Normal Form of Appendix G).

5.1. *Fourier coefficients and their generalization.* — In this section we assume that either:
- *Case (CC)*: $(\mathrm{M}, \beta) = (\mathbf{C} \times \mathbf{C}, dz \wedge dw)$ and we denote by $r(z, w) = -izw$
- or, *Case (AA)*: $(\mathrm{M}, \beta) = (\mathbf{T}_\infty \times \mathbf{C}, d\theta \wedge dr)$ and we denote by $r : (\theta, r) \mapsto r$.

In both cases the flow $t \mapsto \phi^t_{\mathrm{J}\nabla r}$ is 2π-periodic w.r.t. $t \in \mathbf{R}$ (*cf.* (4.84)).

Let U be a connected open set of $\mathbf{C}$ and $\mathrm{F} \in \mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$. For any $m \in \mathrm{W}_{h,\mathrm{U}}$ and any $t \in \mathbf{R}$, $\phi^t_{\mathrm{J}\nabla r}(m) \in \mathrm{W}_{h,\mathrm{U}}$:

$$\begin{cases} (\mathrm{CC}): \ \phi^t_{\mathrm{J}\nabla r}(z, w) = (e^{-it}z, e^{it}w), \\ (\mathrm{AA}): \ \phi^t_{\mathrm{J}\nabla r}(\theta, r) = (\theta + t, r). \end{cases}$$

We can hence define $t \mapsto \mathrm{F}(\phi^t_{\mathrm{J}\nabla r}(m))$ which is a 2π-periodic function $\mathbf{R} \to \mathbf{C}$ and for $n \in \mathbf{Z}$ we introduce its n-th Fourier coefficient $\mathcal{M}_n(\mathrm{F})(m)$:

$$\mathcal{M}_n(\mathrm{F})(m) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\mathrm{F} \circ \phi^t_{\mathrm{J}\nabla r}(m)\,dt \tag{5.99}$$

$$\mathrm{F}(\phi^t_{\mathrm{J}\nabla r}(m)) = \sum_{n \in \mathbf{Z}} \mathcal{M}_n(\mathrm{F})(m)e^{int}. \tag{5.100}$$

The dependence of $\mathcal{M}_n(\mathrm{F})(m)$ is holomorphic in m and we have thus defined $\mathcal{M}_n(\mathrm{F}) \in \mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$. We observe that

$$\mathcal{M}_n(\mathrm{F}) \circ \phi^{2\pi/n}_{\mathrm{J}\nabla r} = \mathcal{M}_n(\mathrm{F}) \tag{5.101}$$

and

$$\forall\, t \in \mathbf{R},\ \mathcal{M}_0(\mathrm{F}) \circ \phi^t_{\mathrm{J}\nabla r} = \mathcal{M}_0(\mathrm{F}).$$

5.1.1. *Case (CC).* — One has

$$\phi^t_{J\nabla r}(z,w) = (e^{-it}z, e^{it}w) \tag{5.102}$$

and if $F = F(z,w)$, (5.99) becomes

$$\mathcal{M}_n(F)(z,w) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}F(e^{-it}z, e^{it}w)dt.$$

If furthermore $F(z,w) = \sum_{(k,l)\in\mathbf{N}^2} F_{k,l}z^k w^l$ is converging on some polydisk $\mathbf{D}(0,\mu)\times\mathbf{D}(0,\mu)$ one has

$$\mathcal{M}_n(F)(z,w) = \sum_{\substack{(k,l)\in\mathbf{N}^2\\ l-k=n}} F_{k,l}z^k w^l \tag{5.103}$$

hence, if for some $p\in\mathbf{N}^*$, $F(z,w) = O^p(z,w)$, then for any $n\in\mathbf{N}$, $\mathcal{M}_n(F)(z,w) = O^p(z,w)$.

5.1.2. *(AA) Case.* — In that case

$$\phi^t_{J\nabla r}(\theta,r) = (\theta+t, r)$$

and if $F = F(\theta,r)$ we define

$$\begin{aligned}\mathcal{M}_n(F)(\theta,r) &= (2\pi)^{-1}\int_0^{2\pi} e^{-int}F(\theta+t,r)dt\\ &= \widehat{F}(n,r)e^{in\theta}\end{aligned}$$

where

$$\widehat{F}(n,r) = (2\pi)^{-1}\int_0^{2\pi} e^{-in\theta}F(\theta,r)d\theta$$

is the n-th Fourier coefficient of $F(\cdot,r)$. Notice that though F is only defined on $\mathbf{T}_h\times U$, $\mathcal{M}_n(F)$ is defined in $\mathbf{T}_\infty\times U$.

Remark **5.1.** — We see from (5.99) that if for some $p>0$, $F = O^p(r)$ (which means that for any $m\in W_{h,U}$ one has $|F(m)|\le C|r(m)|^p$ for some $C>0$) then $\mathcal{M}_nF = O^p(r)$ for any $n\in\mathbf{N}$.

Remark **5.2.** — Using the fact that $\mathcal{M}_n(F)\circ\phi^{2\pi/n}_{J\nabla r} = \mathcal{M}_n(F)$ one can show that $f_F\circ\phi^{2\pi/n}_{J\nabla r} = \phi^{2\pi/n}_{J\nabla r}\circ f_F$ both in the (AA) and (CC) Case.[36]

[36] For example in the (CC)-case, since $\phi^{2\pi/n}_{J\nabla r}(z,w) = (e^{-2\pi i/n}z, e^{2\pi i/n}w)$, the condition on F implies $F(e^{-2\pi i/n}z, e^{2\pi i/n}\widetilde{w}) = F(z,\widetilde{w})$ and the conclusion follows from (4.88).

5.1.3. *Form of* $\mathcal{M}_0(\mathrm{F})$.

Lemma **5.1.** — *If* $\mathrm{F} \in \mathcal{O}_\sigma(\mathrm{W}_{h,\mathrm{U}})$ *there exists* $\mathrm{M}(\mathrm{F}) \in \mathcal{O}_\sigma(\mathrm{U})$ *such that*

$$\mathcal{M}_0(\mathrm{F}) = \mathrm{M}(\mathrm{F}) \circ r, \qquad \|\mathrm{M}(\mathrm{F})\|_{\mathrm{U}} \leq \|\mathrm{F}\|_{h,\mathrm{U}}. \tag{5.104}$$

Moreover

$$f_{\mathrm{M}(\mathrm{F})} = \Phi_{\mathrm{M}(\mathrm{F})} \circ f_{\mathfrak{O}_2(\mathrm{F})}. \tag{5.105}$$

Proof. — By definition of $\mathcal{M}_0(\mathrm{F})$ we see that for every $t \in \mathbf{R}$

$$\mathcal{M}_0(\mathrm{F}) \circ \phi^t_{\mathrm{J}\nabla r} = \mathcal{M}_0(\mathrm{F}).$$

Lemma D.1 of the Appendix provides us with $\mathrm{M}(\mathrm{F}) \in \mathcal{O}_\sigma(\mathrm{U})$ such that $\mathcal{M}_0(\mathrm{F}) = \mathrm{M}(\mathrm{F}) \circ r$. We just have to prove (5.105) in the (CC) case. If $(\widetilde{z}, \widetilde{w}) = f_{\mathrm{M}(\mathrm{F})}(z, w)$ one has

$$\widetilde{z} = (1 + \partial(\mathrm{M}(\mathrm{F}))(z\widetilde{w}))z, \qquad \widetilde{w} = (1 + \widetilde{w}\partial(\mathrm{M}(\mathrm{F}))(z\widetilde{w}))^{-1} w$$

and since $\widetilde{w}(z, w) = w + \mathfrak{O}(\mathrm{F})$ we get

$$(\widetilde{z}, \widetilde{w}) = (e^{-\partial(\mathrm{M}(\mathrm{F}))(zw)} z, e^{\partial(\mathrm{M}(\mathrm{F}))(zw)} w) + \mathfrak{O}_2(\mathrm{F}). \qquad \square$$

5.1.4. *Decay of the* $\mathcal{M}_n(\mathrm{F})$. — We observe that

- in *Case (AA)*, for $m = (\theta, r)$ fixed in $\mathrm{W}_{h,\mathrm{U}}$, the function

$$\begin{cases} \mathbf{T}_{h - |\Im\theta|} \to \mathbf{C} \\ t \mapsto \mathrm{F}(\phi^t_{\mathrm{J}\nabla r}(m)) = \mathrm{F}(\theta + t, r) \end{cases}$$

  is well defined and holomorphic;
- in *Case (CC)*, for $(z, w) \in \mathrm{W}_{h,\mathrm{U}}$ fixed (recall (5.102) and the definition (2.46) of $\mathrm{W}^{\mathrm{CC}}_{h,\mathrm{U}}$), the function

$$\begin{cases} \mathbf{R} + i] - \ln(e^h \rho^{1/2}/|w|), \ln(e^h \rho^{1/2}/|z|)[\to \mathbf{C} \\ t \mapsto \mathrm{F}(\phi^t_{\mathrm{J}\nabla r}(m)) = \mathrm{F}(e^{-it} z, e^{it} w) \end{cases} \tag{5.106}$$

  (with $\rho = \sup\{|r|,\ r \in \mathrm{U}\}$) is also a well defined $2\pi\mathbf{Z}$-periodic holomorphic function. Furthermore, if $m = (z, w) \in \mathrm{W}^{\mathrm{CC}}_{h-\delta,\mathrm{U}}$ one has $\max(|z|, |w|) \leq e^{h-\delta}\rho^{1/2}$ thus $\min(\ln(e^h \rho^{1/2}/|w|), \ln(e^h \rho^{1/2}/|z|)) \geq \delta$.

Hence, in any case, for $m \in \mathrm{W}_{h-\delta,\mathrm{U}}$ the function $t \mapsto \mathrm{F} \circ \phi^t_{\mathrm{J}\nabla r}(m)$ is 2π-periodic, holomorphic on $\mathbf{T}_\delta$ and bounded in module by $\|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}}$. The Fourier coefficients $\mathcal{M}_n(\mathrm{F})(m)$ of the function $t \mapsto \mathrm{F} \circ \phi^t_{\mathrm{J}\nabla r}(m)$

$$\mathrm{F} \circ \phi^t_{\mathrm{J}\nabla r} = \sum_{n \in \mathbf{Z}} e^{int} \mathcal{M}_n(\mathrm{F}) \tag{5.107}$$

thus satisfy

$$\left(\sum_{n\in\mathbf{Z}}|\mathcal{M}_n(\mathrm{F})(m)|^2\right)^{1/2}\leq\|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}}\tag{5.108}$$

and in fact (*cf.* for example [44])

$$\left(\sum_{n\in\mathbf{Z}}e^{2|n|\delta}|\mathcal{M}_n(\mathrm{F})(m)|^2\right)^{1/2}\leq 2^{1/2}\|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}}.\tag{5.109}$$

5.1.5. *Truncations operators.* — Let us define for $\mathrm{N}\in\mathbf{N}\cup\{\infty\}$,

$$\mathrm{T}_{\mathrm{N}}\mathrm{F}=\sum_{|n|<\mathrm{N}}\mathcal{M}_n(\mathrm{F}),\qquad \mathrm{R}_{\mathrm{N}}\mathrm{F}=\mathrm{F}-\mathrm{T}_{\mathrm{N}}\mathrm{F}.$$

Lemma **5.2.** — *If* $\mathrm{F}\in\mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$ *one has*

$$\text{on}\quad \mathrm{W}_{h,\mathrm{U}},\quad \mathrm{F}=\sum_{n\in\mathbf{Z}}\mathcal{M}_n(\mathrm{F})\tag{5.110}$$

$$\|\mathcal{M}_n(\mathrm{F})\|_{\mathrm{W}_{h-\delta,\mathrm{U}}}\lesssim e^{-|n|\delta}\|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}},\tag{5.111}$$

$$\|\mathrm{R}_{\mathrm{N}}\mathrm{F}\|_{\mathrm{W}_{h-\delta,\mathrm{U}}}\lesssim \delta^{-1}e^{-\mathrm{N}\delta}\|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}},\tag{5.112}$$

$$\|\mathrm{T}_{\mathrm{N}}\mathrm{F}\|_{\mathrm{W}_{h-\delta,\mathrm{U}}}\lesssim \|\mathrm{F}\|_{\mathrm{W}_{h,\mathrm{U}}}\qquad (\textit{if}\ \delta^{-1}e^{-\mathrm{N}\delta}\leq 1).\tag{5.113}$$

Furthermore, if for some $p>0$, $\mathrm{F}=\mathrm{O}^p(r)$ *then*

$$\mathrm{R}_{\mathrm{N}}\mathrm{F}=\mathrm{O}^p(r);\tag{5.114}$$

in the (CC Case), if $\mathrm{F}\in\mathcal{O}(\mathrm{W}_{h,\mathrm{U}})\cap\mathrm{O}^3(z,w)$, *one has*

$$(\mathrm{R}_{\mathrm{N}}\mathrm{F})(z,w)=\mathrm{O}^{\mathrm{N}}(z,w).\tag{5.115}$$

Proof. — Inequality (5.111) is a straightforward consequence of (5.109). Equality (5.110) comes from taking $t=0$ in (5.100). (5.112) is a consequence of (5.111) and (5.113) is clear from (5.112). Inequalities (5.114) and (5.115) are consequences respectively of Remark 5.1 and of identity (5.103). □

5.2. *Solution of the truncated cohomological equation.* — We assume that $0<\rho\leq 1$ and that U is a σ-symmetric open connected set of $\mathbf{D}$.

We recall that we have defined in (4.96) (*cf.* Proposition 4.7) for any $\Omega\in\mathcal{O}(\mathrm{U})$ and $\mathrm{Y}\in\mathcal{O}(\mathrm{W}^{\Omega}_{h,\mathrm{U}})$

$$[\Omega]\cdot\mathrm{Y}=\mathrm{Y}\circ\Phi_{\Omega(r)}-\mathrm{Y}.$$

The main Proposition is the following:

Proposition **5.3.** — *Let* $\tau \geq 0$, $\Omega \in \mathcal{O}_\sigma(\mathrm{U})$, $\mathrm{K} > 0$, $\mathrm{N} \in \mathbf{N}^* \cup \{\infty\}$ *be such that one has on* U

$$\forall\, (k,l) \in \mathbf{N}^* \times \mathbf{Z},\ 1 \leq k < \mathrm{N} \implies |k\frac{1}{2\pi}\partial\Omega(\cdot) - l| \geq \mathrm{K}^{-1}|k|^{-\tau}. \tag{5.116}$$

Then, for any $\mathrm{F} \in \mathcal{O}_\sigma(\mathrm{W}_{h,\mathrm{U}})$, *there exists* $\mathrm{Y} \in \mathcal{O}_\sigma(\mathrm{W}^\Omega_{h,\mathrm{U}})$ *such that, on* $\mathrm{W}_{h,\mathrm{U}}$, *one has* $\mathcal{M}_0(\mathrm{Y}) = 0$, $\mathcal{M}_k(\mathrm{Y}) = 0$ *for* $|k| \geq \mathrm{N}$ *and*

$$\mathrm{T_N F} - \mathcal{M}_0(\mathrm{F}) = [\Omega] \cdot \mathrm{Y}. \tag{5.117}$$

This Y *satisfies for any* $0 < \delta < h$

$$\|\mathrm{Y}\|_{\mathrm{W}^\Omega_{h-\delta,\mathrm{U}}} \lesssim \mathrm{K}\min(\delta^{-(1+\tau)}, \mathrm{N}^{\tau+1})\|\mathrm{F}\|_{h,\mathrm{U}}. \tag{5.118}$$

Moreover, if we assume in addition that Ω *is of the form* $\Omega(r) = 2\pi\omega_0 r$, $\omega_0 \in \mathbf{R}$, *then one can improve the exponent in (5.118):*

$$\|\mathrm{Y}\|_{\mathrm{W}^\Omega_{h-\delta,\mathrm{U}}} \lesssim \mathrm{K}\min(\delta^{-\tau}, \mathrm{N}^{\tau})\|\mathrm{F}\|_{h,\mathrm{U}}. \tag{5.119}$$

Proof. — We observe that both in *Case (AA)* or *Case (CC)* one has on $\mathrm{W}_{h,\mathrm{U}} \cap \{r \in \mathbf{R}\}$ (*cf.* (4.84))

$$\Phi_{\Omega(r)} = \phi^{\partial\Omega(r)}_{\mathrm{J}\nabla r}.$$

Hence, if G is a function in $\mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$ one has on $\mathrm{W}_{h,\mathrm{U}} \cap \{r \in \mathbf{R}\}$

$$\begin{aligned}\mathcal{M}_n(\mathrm{G}) \circ \Phi_{\Omega(r)} &= \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\mathrm{G} \circ \phi^{t+\partial\Omega(r)}_{\mathrm{J}\nabla r}dt \\ &= e^{in\partial\Omega(r)}\frac{1}{2\pi}\int_0^{2\pi} e^{-int}\mathrm{G} \circ \phi^{t}_{\mathrm{J}\nabla r}dt \\ &= e^{in\partial\Omega(r)}\mathcal{M}_n(\mathrm{G})\end{aligned}$$

and since $\mathcal{M}_n(\mathrm{G}) \in \mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$, the left hand side of the preceding equations can be holomorphically extended to a function in $\mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$. We then have in $\mathcal{O}(\mathrm{W}_{h,\mathrm{U}})$

$$[\Omega] \cdot \mathcal{M}_n(\mathrm{G}) = (e^{in\partial\Omega(r)} - 1)\mathcal{M}_n(\mathrm{G}).$$

Note that from Lemma M.1 one has for $r \in \mathrm{U}$, $|e^{in\partial\Omega(r)} - 1| \geq \mathrm{K}^{-1}|n|^{-\tau}$. If we define Y by

$$\mathrm{Y} = \sum_{0<|n|<\mathrm{N}} \frac{1}{e^{in\partial\Omega(r)} - 1}\mathcal{M}_n(\mathrm{F}) \tag{5.120}$$

we have from Lemma 5.2

$$\|Y\|_{W_{h-\delta,U}} \lesssim K \sum_{1\le|n|<N} |n|^{\tau} e^{-|n|\delta} \|F\|_{h,U}$$
$$\lesssim \min(K\delta^{-(1+\tau)}, KN^{\tau+1})\|F\|_{h,U}$$

and

$$[\Omega]\cdot Y = Y\circ\Phi_{\Omega} - Y = T_N F - \mathcal{M}_0(F).$$

This last formula shows that if we define $\widetilde{Y}$ on $\Phi_{\Omega(r)}(W_{h-\delta,U})$ by $\widetilde{Y}\circ\Phi_{\Omega(r)} = T_N F - \mathcal{M}_0(F) + Y$ the functions $\widetilde{Y}$ and Y coincide on $\Phi_{\Omega(r)}(W_{h-\delta,U})\cap W_{h-\delta,U}$ and thus Y can be holomorphically extended to $\Phi_{\Omega(r)}(W_{h-\delta,U})\cup W_{h-\delta,U} =: W^{\Omega}_{h-\delta,U}$ and

$$\|Y\|_{W^{\Omega}_{h-\delta,U}} \lesssim \min(K\delta^{-(1+\tau)}, KN^{\tau+1})\|F\|_{h,U},$$

which is (5.118).

The fact that $\mathcal{M}_0(Y)=0$ and its uniqueness (under the condition $\mathcal{M}_0(Y)=0$) comes again from Lemma 5.2. Finally, the σ-symmetry of Y on $W_{h,U}$ is clear.

To conclude the proof of the Proposition we just have to check that if $(2\pi)^{-1}\Omega(r)\equiv\omega_0\in\mathbf{R}$ satisfies (5.116) then (5.119) holds. This is a result due to Rüssmann [45] that we now recall for completeness. In fact, Rüssmann proves that if

$$D_n = \min_{l\in\mathbf{Z}} |n\omega_0 - l|, \qquad D_n^* = \min_{1\le j\le n} D_j$$

one has

$$\sum_{n=1}^{N} D_n^{-2} \le (\pi^2/3)(D_N^*)^{-2}. \tag{5.121}$$

From Lemma M.1 one has $|e^{2\pi i n\omega_0}-1| \ge 4\min_{l\in\mathbf{Z}}|n\omega_0 - l| = 4D_n$. Thus, if we apply Cauchy-Schwarz inequality to (5.120) we have for $\nu=0$ or $\nu=\delta$

$$\|Y\|_{W_{h-\delta,U}} \lesssim \left(\sum_{1\le|n|<N} e^{-2|n|\nu}D_n^{-2}\right)^{1/2}\left(\sum_{1\le|n|<N} e^{2|n|\nu}\|\mathcal{M}_n(F)\|^2_{W_{h-\delta,U}}\right)^{1/2}$$
$$\lesssim \left(\sum_{1\le|n|<N} e^{-2|n|\nu}D_n^{-2}\right)^{1/2}\|F\|_{W_{h,U}} \qquad (cf.\ (5.108), (5.109)).$$

– If $N<\infty$, we take $\nu=0$ and (5.121) gives

$$\|Y\|_{W_{h-\delta,U}} \lesssim (D_N^*)^{-1}\|F\|_{h,U}$$
$$\lesssim KN^{\tau}\|F\|_{h,U}.$$

– If $N=\infty$ we take $\nu=\delta$. Taking into account (5.121), we perform an Abel summation (discrete integration by part) on the sums $\sum_{1\le n<N} e^{-2|n|\nu}D_n^{-2}$, $\sum_{1\le -n<N} e^{-2|n|\nu}D_n^{-2}$: this yields

$$\begin{aligned}\|Y\|_{W_{h-\delta,U}} &\lesssim \left(\sum_{1\le|n|<\infty}(e^{-2|n|\delta}-e^{-2(|n|+1)\delta}(D_n^*)^{-2}\right)^{1/2}\|F\|_{h,U}\\ &\lesssim \left(\sum_{1\le|n|<\infty}\delta e^{-2|n|\delta}K^2|n|^{2\tau}\right)^{1/2}\|F\|_{h,U}\\ &\lesssim K\delta^{-\tau}\|F\|_{h,U}.\end{aligned}$$

□

Remark **5.3.** — If in Proposition 5.3 $U=\mathbf{D}(0,\rho)$ is a disk centered at 0 and

$$\begin{cases}\text{(AA)-case } F(\theta,r)=\sum_{k\in\mathbf{N}}\sum_{l\in\mathbf{Z}}\widehat{F}_k(l)e^{il\theta}r^k\\ \text{(CC)-case } F(z,w)=\sum_{(k,l)\in\mathbf{N}}F_{k,l}z^kw^l\end{cases}$$

one has the more explicit expressions

$$\begin{cases}\text{(AA)-case } Y(\theta,r)=\sum_{k\in\mathbf{N}}\sum_{l\in\mathbf{Z}^*}\dfrac{\widehat{F}_k(l)}{e^{il\partial\Omega(r)}-1}e^{il\theta}r^k\\ \text{(CC)-case } Y(z,w)=\sum_{\substack{(k,l)\in\mathbf{N}\\ l\neq k}}\dfrac{F_{k,l}}{e^{i(l-k)\partial\Omega(r)}-1}z^kw^l.\end{cases}\tag{5.122}$$

In particular, if

$$\Omega(r)=2\pi\omega_0 r \qquad\text{and}\qquad \begin{cases}\text{(CC)-case} & F(z,w)=O^m(z,w)\\ \text{(AA)-case} & F(\theta,r)=O^m(r)\end{cases}$$

then Y satisfies also (see the remarks at the end of Sections 5.1.1 and 5.1.2)

$$\begin{cases}\text{(CC)-case} & Y(z,w)=O^m(z,w)\\ \text{(AA)-case} & Y(\theta,r)=O^m(r).\end{cases}$$

5.3. *Fundamental conjugation step.* — We begin by the following consequence of Proposition 4.7. Let U be a holed domain, $h>0$.

Lemma **5.4.** — *There exists* $\bar{a}\ge 2$ *and* $C>0$ *such that if* $\Omega\in\mathcal{O}_\sigma(U)$, $F\in\mathcal{O}_\sigma(W_{h,U})$, $Y\in\mathcal{O}_\sigma(W^{\Omega}_{h,U})$ *and* $\delta=\mathfrak{d}(F,Y;W_{h,U})$, $\delta>0$ *satisfies*

$$C\,\underline{d}(W_{h,U})^{-\bar{a}}\delta^{-\bar{a}}\|F\|_{W_{h,U}}\le 1\tag{5.123}$$

then one has on $e^{-\delta}W_{h,U}$ *(cf. Lemma 5.1 for the definition of* $M(F)$*)*

$$f_Y \circ \Phi_\Omega \circ f_F \circ f_Y^{-1} = \Phi_{\Omega+M(F)} \circ f_{F-\mathcal{M}_0(F)+[\Omega+M(F)]\cdot Y+\dot{\mathfrak{O}}_2^{(\overline{a})}(Y,F)}. \tag{5.124}$$

Proof. — We first observe that since $F = F - \mathcal{M}_0(F) + \mathcal{M}_0(F)$, we have by (4.94) and Lemma 5.1

$$\begin{aligned} f_F &= f_{\mathcal{M}_0(F)} \circ f_{F-\mathcal{M}_0(F)+\mathfrak{O}_2(F)} \\ &= \Phi_{M(F)} \circ f_{\mathfrak{O}_2(F)} \circ f_{F-\mathcal{M}_0(F)+\mathfrak{O}_2(F)} \\ &= \Phi_{M(F)} \circ f_{F-\mathcal{M}_0(F)+\mathfrak{O}_2(F)} \end{aligned}$$

and thus

$$\Phi_\Omega \circ f_F = \Phi_{\Omega+M(F)} \circ f_{F-\mathcal{M}_0(F)+\mathfrak{O}_2(F)}.$$

Now we use Proposition 4.7 and make explicit the notations $\mathfrak{d}$ and $\dot{\mathfrak{O}}$: for some $\overline{a} > 0$ that we can choose ≥ 2 and some $C > 0$, if (5.123) is satisfied, one has

$$\begin{aligned} &f_Y \circ \Phi_{\Omega+M(F)} \circ f_{F-\mathcal{M}_0(F)+\mathfrak{O}_2(F)} \circ f_Y^{-1} \\ &\quad = \Phi_{\Omega+M(F)} \circ f_{F-\mathcal{M}_0(F)+[\Omega+M(F)]\cdot Y+\dot{\mathfrak{O}}_2^{(\overline{a})}(Y,F)}. \end{aligned} \tag{5.125}$$

□

Proposition **5.5.** — *Let* $\overline{a}_0 = \overline{a} + 4$ *(*$\overline{a}$ *from Lemma 5.4). There exists* $\overline{C} > 0$ *such that the following holds. Let* U *be a holed domain,* $\Omega \in \mathcal{O}_\sigma(U)$*, and* $F \in \mathcal{O}_\sigma(W_{h,U})$*. Assume that there exists a holed domain* $V \subset U$*,* $N \in \mathbf{N}^* \cup \{\infty\}$ *and* $K > 0$ *such that on* V *the following non-resonance condition (cf. (5.116)) is satisfied:*

$$\forall\ (k,l) \in \mathbf{N}^* \times \mathbf{Z},\ \ 1 \leq k < N \implies |k\frac{1}{2\pi}\partial\Omega(\cdot) - l| \geq K^{-1}|k|^{-\tau} \tag{5.126}$$

and assume that $\overline{C}N^{-1} < \delta < \min(h, \overline{C}^{-1})$ *is such that* $e^{-\delta}W_{h,V}$ *is not empty and*

$$(\delta\, \underline{d}(W_{h,V}))^{-(\overline{a}_0+\tau)} K \|F\|_{h,U} < \overline{C}^{-1}. \tag{5.127}$$

Then there exists $Y \in \mathcal{O}(W^\Omega_{h,V})$ *solution on* $W^\Omega_{h,V}$ *of the cohomological equation (cf. (5.117), (5.118)):*

$$T_N F - \mathcal{M}_0(F) = -[\Omega]\cdot Y, \qquad \|Y\|_{e^{-\delta/2}W^\Omega_{h,V}} \lesssim K\delta^{-(1+\tau)}\|F\|_{W_{h,U}} \tag{5.128}$$

and $\widetilde{\Omega} \in \mathcal{O}(e^{-\delta}W_{h,V})$*,* $\widetilde{F} \in \mathcal{O}_\sigma(W_{h,U})$ *such that one has on* $e^{-\delta}W_{h,V}$

$$\begin{aligned} &f_Y \circ \Phi_{\Omega(r)} \circ f_F \circ f_Y^{-1} = \Phi_{\widetilde{\Omega}(r)} \circ f_{\widetilde{F}}, \qquad \widetilde{\Omega} = \Omega + M(F) \\ &\|\widetilde{F}\|_{C^3(e^{-\delta}W_{h,V})} \leq K(\delta\,\underline{d}(W_{h,V}))^{-(\overline{a}_0+\tau)}\Big(\|F\|^2_{h,U} + e^{-N\delta/2}\|F\|_{h,U}\Big). \end{aligned} \tag{5.129}$$

Proof. — We apply Proposition 5.3 to obtain some Y satisfying (5.128) and we apply Lemma 5.4 with δ equal to $\delta/2$. Since (*cf.* (4.97)) $[\Omega + \mathrm{M}(\mathrm{F})] \cdot \mathrm{Y} = [\Omega] \cdot \mathrm{Y} + \mathrm{O}(|\nabla \mathrm{Y}||\nabla(\mathrm{M}(\mathrm{F}))|) = [\Omega] \cdot \mathrm{Y} + \mathfrak{O}_2^{(2)}(\mathrm{Y}, \mathrm{F})$, we get using $[\Omega] \cdot \mathrm{Y} + \mathrm{F} - \mathcal{M}_0(\mathrm{F}) = \mathrm{R}_\mathrm{N}\mathrm{F}$ (*cf.* (5.117)),

$$e^{-\delta/2}\mathrm{W}_{h,\mathrm{V}}, \qquad f_\mathrm{Y} \circ \Phi_\Omega \circ f_\mathrm{F} \circ f_\mathrm{Y}^{-1} =: \Phi_{\widetilde{\Omega}} \circ f_{\widetilde{\mathrm{F}}}$$

with

$$\widetilde{\Omega} = \Omega + \mathrm{M}(\mathrm{F}) \tag{5.130}$$

$$\widetilde{\mathrm{F}} = \mathrm{R}_\mathrm{N}\mathrm{F} + \mathfrak{O}_2^{(\overline{a})}(\mathrm{Y}, \mathrm{F}). \tag{5.131}$$

The definition of the symbol $\mathfrak{O}_2^{(\overline{a})}$, (5.112) and (5.128) show that there exists a universal positive constant $\overline{\mathrm{C}}$ such that if (5.127) is satisfied one has

$$\|\widetilde{\mathrm{F}}\|_{e^{-\delta/2}\mathrm{W}_{h,\mathrm{V}}} \lesssim \mathrm{K}\delta^{-(1+\tau)}(\delta^{-1}\,\underline{\mathrm{d}}(\mathrm{W}_{h,\mathrm{V}})^{-1})^{\overline{a}}\|\mathrm{F}\|_{h,\mathrm{U}}^2 + \delta^{-1}e^{-\mathrm{N}\delta/2}\|\mathrm{F}\|_{h,\mathrm{U}}. \tag{5.132}$$

Inequalities (5.129), comes from (5.132) and Cauchy's inequality (2.53) of Section 2.3.4 (applied with $e^{-\delta}\mathrm{W}_{h,\mathrm{V}}$ and $e^{-\delta/2}\mathrm{W}_{h,\mathrm{V}}$ in place of $e^{-\delta}\mathrm{W}_{h,\mathrm{U}}$ and $\mathrm{W}_{h,\mathrm{U}}$) because $\underline{\mathrm{d}}(e^{-\delta/2}\mathrm{W}_{h,\mathrm{V}}) \geq (1/2)\,\underline{\mathrm{d}}(\mathrm{W}_{h,\mathrm{V}})$ if $\delta < 1/10$ (which is the case if $\overline{\mathrm{C}}$ is large enough). □

6. Birkhoff Normal Forms

6.1. *Formal Normal Forms.* — We recall in this subsection the classical results on (formal) Birkhoff Normal Forms. For more details on the related formal aspects we refer to Appendix E. We also explain how Pérez-Marco's dichotomy extends to the diffeomorphism case (in particular in the (AA)-case).

6.1.1. *BNF near a non-resonant elliptic fixed point ((CC) case).* — Let $\widetilde{f} : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be a real analytic symplectic diffeomorphism of the form $\widetilde{f}(x, y) = \mathrm{D}\widetilde{f}(0, 0) \cdot (x, y) + \mathrm{O}^2(x, y)$ where

$$\mathrm{D}\widetilde{f}(0, 0) = \Phi_{2\pi\omega_0 r} = \begin{pmatrix} \cos(2\pi\omega_0) & -\sin(2\pi\omega_0) \\ \sin(2\pi\omega_0) & \cos(2\pi\omega_0) \end{pmatrix}$$

with $\omega_0 \in \mathbf{R} \setminus \mathbf{Q}$.

If $\varphi : \mathbf{C}^2 \to \mathbf{C}^2$ is the change of coordinates $\varphi(x, y) = (z, w)$ defined in (4.77) the diffeomorphism $f := \varphi \circ \widetilde{f} \circ \varphi^{-1}$ is exact symplectic and of the form $f(z, w) = \Phi_{2\pi\omega_0 r}(z, w) + \mathrm{O}^2(z, w)$ where $r(z, w) = -izw$

$$\Phi_{2\pi\omega_0 r}(z, w) = (e^{-2\pi i\omega_0}z, e^{2\pi i\omega_0}w).$$

From Lemma 4.5 we have the representation

$$f = \Phi_{2\pi\omega_0 r} \circ f_F, \qquad F = O^3(z, w)$$

for some $F \in \mathcal{O}_\sigma(\mathbf{D}(0,\mu)^2)$, $\mu > 0$. We then have the following classical proposition that establishes the existence of Birkhoff Normal Forms to arbitrarily high order.

Proposition **6.1.** — *Let* $\omega_0 \in \mathbf{R} \setminus \mathbf{Q}$. *Then, for any* $N \geq 3$ *there exist* σ*-symmetric holomorphic maps* $\Omega_N : (\mathbf{C}, 0) \to \mathbf{C}$, $Z_N, F_N : (\mathbf{C}^2, 0) \to \mathbf{C}$ *such that on a neighborhood of* $0 \in \mathbf{C}^2$ *one has* ($r = -izw$)

$$(\mathbf{6.133}) \qquad \begin{cases} f_{Z_N} \circ (\Phi_{2\pi\omega_0 r} \circ f_F) \circ f_{Z_N}^{-1} = \Phi_{\Omega_N} \circ f_{F_N} \\ F_N(z, w) = O^{2(N+1)}(z, w), \quad Z_N(z, w) = O^3(z, w), \\ \quad \Omega_N(r) = 2\pi\omega_0 r + O^2(r). \end{cases}$$

Remark **6.1.** — The sequences $(Z_N)_N$ and $(\Omega_N)_N$ converge respectively in $\mathbf{C}[[z, w]]$ and in $\mathbf{R}[[r]]$. If $Z_\infty \in \mathbf{C}[[z, w]]$ and $\Omega_\infty \in \mathbf{R}[[r]]$ are their respective limits one has in $\mathbf{C}[[z, w]]$ the formal identity

$$(\mathbf{6.134}) \qquad \begin{cases} f_{Z_\infty} \circ (\Phi_{2\pi\omega_0 r} \circ f_F) \circ f_{Z_\infty}^{-1} = \Phi_{\Omega_\infty} \\ Z_\infty(z, w) = O^3(z, w), \qquad \Omega_\infty(r) = 2\pi\omega_0 r + O^2(r). \end{cases}$$

Conversely, (6.134) defines Ω_∞ uniquely;[37] Ω_∞ is the Birkhoff Normal Form BNF(f) of f (and BNF($\tilde{f}$) of $\tilde{f}$). In particular, BNF(f) is invariant by (formal) symplectic conjugacies which are tangent to the identity.

Remark **6.2.** — If $f = \Phi_\Omega \circ f_F$ with $\Omega = \Omega(r) = 2\pi\omega_0 r + O(r^2)$ and $F(z, w) = O^{2(N+1)}(z, w)$ then

$$(\mathbf{6.135}) \qquad \mathrm{BNF}(f)(r) = \Omega(r) + O^{N+1}(r).$$

6.1.2. *BNF near a KAM circle (Action-Angle case).* — Let $f : (\mathbf{T} \times \mathbf{R}, \mathbf{T} \times \{0\}) \to (\mathbf{T} \times \mathbf{R}, \mathbf{T} \times \{0\})$ be a real analytic symplectic diffeomorphism of the form $f(\theta, r) = (\theta + 2\pi\omega_0, r) + (O(r), O(r^2))$. We notice that $\Phi_{2\pi\omega_0 r} : (\theta, r) \mapsto (\theta + 2\pi\omega_0, r)$. We can thus write f under the form ($h, \rho > 0$)

$$f = \Phi_{2\pi\omega_0 r} \circ f_F, \qquad F \in \mathcal{O}_\sigma(e^{2h}(\mathbf{T}_h \times \mathbf{D}(0, \rho))), \qquad F = O^2(r).$$

[37] The normalizing map Z_∞ is unique up to composition on the left by a formal generalized symplectic rotation Φ_A, $A \in \mathbf{R}[[r]]$.

Proposition **6.2.** — *Let* $\omega_0 \in \mathbf{R}$ *be Diophantine. Then, for any* $\mathrm{N} \geq 3$ *there exist real analytic maps* $\Omega_{\mathrm{N}} : (\mathbf{R}, 0) \to \mathbf{R}$, $\mathrm{Z}_{\mathrm{N}}, \mathrm{F}_{\mathrm{N}} : (\mathbf{T} \times \mathbf{R}, \mathbf{T} \times \{0\}) \to \mathbf{R}$ *such that*

$$(\mathbf{6.136}) \qquad \begin{cases} f_{\mathrm{Z}_{\mathrm{N}}} \circ (\Phi_{2\pi\omega_0 r} \circ f_{\mathrm{F}}) \circ f_{\mathrm{Z}_{\mathrm{N}}}^{-1} = \Phi_{\Omega_{\mathrm{N}}} \circ f_{\mathrm{F}_{\mathrm{N}}} \\ \mathrm{F}_{\mathrm{N}}(\theta, r) = \mathrm{O}^{\mathrm{N}+1}(r), \quad \mathrm{Z}_{\mathrm{N}}(\theta, r) = \mathrm{O}^2(r), \quad \Omega_{\mathrm{N}}(r) = 2\pi\omega_0 r + \mathrm{O}^2(r). \end{cases}$$

Remark **6.3.** — Let $\mathrm{C}^\omega(\mathbf{T})[[r]]$ (where $\mathrm{C}^\omega(\mathbf{T}) = \bigcup_{h>0} \mathrm{C}^\omega_h(\mathbf{T})$) be the set of formal power series

$$(\mathbf{6.137}) \qquad \mathrm{F}(\theta, r) = \sum_{n \in \mathbf{N}} \mathrm{F}_n(\theta) r^n, \qquad \mathrm{F}_n \in \mathrm{C}^\omega(\mathbf{T}) \text{ for all } n \in \mathbf{N}.$$

The sequence $(\mathrm{Z}_{\mathrm{N}})_{\mathrm{N}}$ converges in $\mathrm{C}^\omega(\mathbf{T})[[r]]$ and the sequence $(\Omega_{\mathrm{N}})_{\mathrm{N}}$ converges in $\mathbf{R}[[r]]$. If $\mathrm{Z}_\infty \in \mathrm{C}^\omega(\mathbf{T})[[r]]$ and $\Omega_\infty \in \mathbf{R}[[r]]$ are their respective limits one has in $\mathrm{C}^\omega(\mathbf{T})[[r]]$ the formal identity

$$(\mathbf{6.138}) \qquad \begin{cases} f_{\mathrm{Z}_\infty} \circ (\Phi_{2\pi\omega_0 r} \circ f_{\mathrm{F}}) \circ f_{\mathrm{Z}_\infty}^{-1} = \Phi_{\Omega_\infty} \\ \mathrm{Z}_\infty(\theta, r) = \mathrm{O}^2(r), \qquad \Omega_\infty(r) = 2\pi\omega_0 r + \mathrm{O}^2(r). \end{cases}$$

Conversely, (6.138) defines Ω_∞ uniquely;[38] Ω_∞ is the Birkhoff Normal Form $\mathrm{BNF}(f)$ of f. In particular, $\mathrm{BNF}(f)$ is invariant by (formal) symplectic conjugacies which are of the form $id + (\mathrm{O}(r), \mathrm{O}(r^2))$.

Remark **6.4.** — If $f = \Phi_\Omega \circ f_{\mathrm{F}}$ with $\Omega = \Omega(r) = 2\pi\omega_0 r + \mathrm{O}(r^2)$ and $\mathrm{F}(\theta, r) = \mathrm{O}^{\mathrm{N}+1}(r)$ then

$$(\mathbf{6.139}) \qquad \mathrm{BNF}(f)(r) = \Omega(r) + \mathrm{O}^{\mathrm{N}+1}(r).$$

Remark **6.5.** — The reason why we impose a Diophantine condition on ω_0 in the statement of Proposition 6.2 is the following. The existence of the formal Birkhoff Normal Form (6.136) derives from an inductive procedure where at each step $n \in \mathbf{N}^*$ one constructs a formal conjugation f_{Y_n} with $\mathrm{Y}_n \in \mathrm{C}^\omega(\mathbf{T})[[r]]$ that conjugates f_{F_n} ($\mathrm{F}_n \in \mathrm{C}^\omega(\mathbf{T})[[r]]$, $\mathrm{F}_n = \mathrm{O}^n(r)$) to $f_{\mathrm{F}_{n+1}}$ ($\mathrm{F}_{n+1} \in \mathrm{C}^\omega(\mathbf{T})[[r]]$, $\mathrm{F}_{n+1}(\theta, r) = \mathrm{O}^{n+1}(r)$). To perform this conjugation step one has to solve a cohomological equation $\mathrm{F}_n(\theta, r) = \mathrm{Y}_n(\theta + 2\pi\omega_0, r) - \mathrm{Y}_n(\theta, r) + \int_0^{2\pi} \mathrm{F}_n(\varphi, r) d\varphi$ where r is a formal variable but θ lies on $\mathbf{T}$ (see Lemma E.7). This equation is classically solved by passing to Fourier coefficients (see for example [13]) but it involves *small denominators* that can be dealt with if ω_0 satisfies an arithmetic condition, for example a Diophantine one (weaker conditions such as Bruno condition or even $\ln q_{n+1} = o(q_n)$ will also be fine[39]).

[38] The normalizing map Z_∞ is unique up to composition on the left by a formal integrable twist of the form Φ_{A}, $\mathrm{A} \in \mathbf{R}[[r]]$.

[39] As usual p_n/q_n are the convergents of ω_0.

6.2. *Pérez-Marco's Dichotomy.* — We now discuss the extension of Pérez-Marco's Dichotomy, Theorem 1.3, to the diffeomorphism setting.

The first part of Pérez-Marco's argument in [36], translated in our (CC)-setting, is based on the fact that the coefficients of the Birkhoff Normal Form $B(r) = \sum_{n\in\mathbf{N}^d} b_n(F)r^n = \sum_{n\in\mathbf{N}^d} b_n(F)(-izw)^n$ of $\Phi_{2\pi\langle\omega,r\rangle}\circ f_F$ depend polynomially on the coefficients of $F(z,w) = \sum_{(k,l)\in\mathbf{N}^d\times\mathbf{N}^d} F_{k,l}z^k w^l$. More precisely, if we denote by $[F]_j$, $j\geq 3$, the homogeneous part of F of degree j, $[F]_j = \sum_{|k|+|l|=j} F_{k,l}z^k w^l$, then the coefficients of the homogeneous part of degree $2j$, $[B\circ r]_{2j} = \sum_{|k|=j} b_k(F)(-izw)^k$ of $B\circ r$, are polynomials of degree $2j-2$ in the coefficients of $[F]_3,\ldots [F]_j$. As a consequence, if $(z,w)\mapsto F(z,w)$, $(z,w)\mapsto G(z,w)$ are two σ-symmetric holomorphic functions such that $F(z,w) = O^3(z,w)$, $G(z,w) = O^3(z,w)$, then for any $n\geq 3$, the maps $t\mapsto b_n(tF+(1-t)G)$ are polynomials of degree $\leq 2|n|-2$. The second argument in [36] is then to use results from potential theory (in particular the Bernstein-Walsh Lemma[40]) applied to the family of polynomials $t\mapsto b_n(tF+(1-t)G)$ that have a degree which behaves linearly in n.

To check that the arguments of [36] adapt to the diffeomorphism case it is hence enough to check that $t\mapsto b_n(tF+(1-t)G)$ are polynomials of degree $\leq 2(|n|-1)$:

Lemma **6.3.** — *If* F, G *are* σ*-symmetric holomorphic maps* $F, G = O^3(z,w)$ *in the (CC)-case (resp.* $F, G = O^2(r)$ *in the (AA)-case) then, for every* $n\in\mathbf{N}^d$, $|n|\geq 2$, $t\mapsto b_n(tF+(1-t)G)$ *is a polynomial of degree* $\leq 2(|n|-1)$ *(resp.* $\leq |n|-1$*).*

Proof. — We refer to Appendix E where we discuss formal aspects of the BNF (mainly in the (AA)-case) and give a proof of the lemma in Section E.3. ☐

6.3. *Approximate BNF.*

6.3.1. *Elliptic fixed point case ((CC)-Case).* — Our aim is to give a more quantitative version of Proposition 6.1.

Recall that $W_{h,\mathbf{D}(0,\rho)} = \{(z,w)\in\mathbf{D}(0,e^h\rho^{1/2})^2,\ -izw\in\mathbf{D}(0,\rho)\}$ and we denote sometimes by $W_{h,\rho}$ the set $W_{h,\mathbf{D}(0,\rho)}$.

Let $m\geq 4$ be an integer. Applying Proposition 6.1 with $m = N-1$ we can assume that the diffeomorphism f is of the form

$$\begin{cases} f = \Phi_{\Omega_0}\circ f_{F_0} \\ \Omega_0(r) = 2\pi\omega_0 r + O^2(r), \quad \text{and } F_0(z,w) = O^{2m}(z,w). \end{cases} \tag{6.140}$$

In particular (*cf.* Remark 2.1) for some $h>0$ and any $\rho>0$ small enough we can assume that

$$\|F\|_{e^h W_{h,\mathbf{D}(0,\rho)}} \lesssim \rho^m. \tag{6.141}$$

[40] It states that if a polynomial of degree n is bounded above by some constant M on a not pluripolar compact set $K\subset\mathbf{C}^m$ then its size at any point $z\in\mathbf{C}^m$ is not larger than $M\times\exp(ng_K(z))$ where $g_K(z)$ is the Green function of K with pole at ∞.

Denote by $(p_n/q_n)_{n\geq 1}$ the sequence of best rational approximations of ω_0 which has the following properties (*cf.* [20], Chap. 5, formulae (7.3.1)–(7.3.2) and Prop. 7.4): for all $n\in\mathbf{N}^*$

$$\frac{1}{q_n+q_{n+1}}<(-1)^n(q_n\omega_0-p_n)<\frac{1}{q_{n+1}}, \tag{6.142}$$

and

$$\forall\ 0<k<q_n,\ \forall\ l\in\mathbf{Z},\ \ |k\omega_0-l|\geq|q_{n-1}\alpha-p_{n-1}|>\frac{1}{2q_n}. \tag{6.143}$$

We refer to Notations 2.3, 2.6 and 4.8 before stating the following proposition.

Proposition **6.4.** — *Let* $\overline{a}_1:=\max(2\overline{a}+1,30)$ *where* $\overline{a}$ *is the exponent that appears in Lemma 5.4 and assume that (6.141) holds for some* $m\geq\overline{a}_1$. *Then for any* $\beta>0$ *and any* $n\gg_\beta 1$ *there exist* $g^{\mathrm{BNF}}_{q_n^{-1}}\in\widetilde{\mathrm{Symp}}_{ex.,\sigma}(\mathrm{W}_{h,q_n^{-6}})$, *and functions* $\mathrm{F}^{\mathrm{BNF}}_{q_n^{-1}}\in\mathcal{O}_\sigma(\mathrm{W}_{h,q_n^{-6}})\cap\mathrm{O}^{q_n^{1-\beta}}(z,w)$, $\Omega^{\mathrm{BNF}}_{q_n^{-1}}\in\widetilde{\mathcal{O}}_\sigma(\mathbf{D}(0,q_n^{-6}))$ *such that*

$$[\mathrm{W}_{h,q_n^{-6}}]\qquad (g^{\mathrm{BNF}}_{q_n^{-1}})^{-1}\circ\Phi_{\Omega_0}\circ f_{\mathrm{F}_0}\circ g^{\mathrm{BNF}}_{q_n^{-1}}=\Phi_{\Omega^{\mathrm{BNF}}_{q_n^{-1}}}\circ f_{\mathrm{F}^{\mathrm{BNF}}_{q_n^{-1}}} \tag{6.144}$$

$$\Omega^{\mathrm{BNF}}_{q_n^{-1}}(r)-\mathrm{BNF}(f)(r)=\mathrm{O}^{q_n^{1-\beta}}(r),\ \ \textit{in}\ \mathbf{R}[[r]] \tag{6.145}$$

$$\|\Omega^{\mathrm{BNF}}_{q_n^{-1}}\|_{\mathrm{C}^3}\lesssim 1 \tag{6.146}$$

$$\|g^{\mathrm{BNF}}_{q_n^{-1}}-id\|_{\mathrm{C}^1}\leq q_n^{-(m-27)} \tag{6.147}$$

$$\|\mathrm{F}^{\mathrm{BNF}}_{q_n^{-1}}\|_{\mathrm{W}_{h,q_n^{-6}}}\leq\exp(-q_n^{1-\beta}). \tag{6.148}$$

If $\Omega\in\mathcal{TC}(\mathrm{A},\mathrm{B})$ *(see Notation 2.6) one can choose* $\Omega^{\mathrm{BNF}}_{q_n^{-1}}\in\mathcal{TC}(2\mathrm{A},2\mathrm{B})$.

Proof. — See the Appendix, Section F.2. □

6.3.2. *(AA) or (CC) case when* ω_0 *is Diophantine.* — We formulate here a more quantitative version of the classical Birkhoff Normal Form Theorem (Propositions 6.1, 6.2) which holds both in the (AA) or (CC) cases, provided ω_0 is Diophantine:

$$\forall\ k\in\mathbf{Z}\setminus\{0\},\ \min_{l\in\mathbf{Z}}|k\omega_0-l|\geq\frac{\kappa}{|k|^\tau}\qquad(\tau\geq 1). \tag{6.149}$$

Let as usual $\mathrm{W}_{h,\mathbf{D}(0,\rho)}$ be equal to either $\mathrm{W}^{\mathrm{CC}}_{h,\mathbf{D}(0,\rho)}$ or $\mathrm{W}^{\mathrm{AA}}_{h,\mathbf{D}(0,\rho)}$ and $\Omega\in\mathcal{O}_\sigma(\mathbf{D}(0,1))$, $\Omega(r)=2\pi\omega_0 r+\mathrm{O}(r^2)$, where ω_0 is assumed to be Diophantine with exponent τ.

We define (as before $\overline{a}$ is the constant introduced in Lemma 5.4)

$$\overline{a}_{1,\tau}:=\max(2(\tau+\overline{a}),12) \tag{6.150}$$

and we assume that for some $m \geq \bar{a}_{1,\tau}$, the function $\mathrm{F} \in \mathcal{O}_\sigma(e^h \mathrm{W}_{h,\mathbf{D}(0,1/2)})$ $(h>0)$ satisfies

$$\textbf{(6.151)} \qquad \begin{cases} \text{(CC)} - \mathit{Case}: & \mathrm{F}(z,w) = \mathrm{O}^{2m}(z,w) \\ \text{(AA)} - \mathit{Case}: & \mathrm{F}(\theta, r) = \mathrm{O}(r^m). \end{cases}$$

We set

$$\textbf{(6.152)} \qquad \begin{cases} \text{(CC)} - \mathit{Case}: & b_\tau = 2(\tau+1) \\ \text{(AA)} - \mathit{Case}: & b_\tau = \tau + 1. \end{cases}$$

Proposition **6.5.** — *Assume that ω_0 satisfies (6.149) and that for some $m \geq \bar{a}_{1,\tau}$ (6.151) holds. Then, for any $\beta > 0$ and any $0 < \rho \ll_\beta 1$, there exist $\Omega_\rho^{\mathrm{BNF}} \in \widetilde{\mathcal{O}}_\sigma(\mathbf{D}(0,\rho^{b_\tau}))$, $\mathrm{F}_\rho^{\mathrm{BNF}} \in \mathcal{O}_\sigma(\mathrm{W}_{h,\mathbf{D}(0,\rho^{b_\tau})}) \cap \mathrm{O}^{(1/\rho)^{1-\beta}}(r)$ and $g_\rho^{\mathrm{BNF}} \in \widetilde{\mathrm{Symp}}_{ex.,\sigma}(\mathrm{W}_{h,\mathbf{D}(0,\rho^{b_\tau})})$ such that on $\mathrm{W}_{h,\mathbf{D}(0,\rho^{b_\tau})}$ one has*

$$\textbf{(6.153)} \qquad (g_\rho^{\mathrm{BNF}})^{-1} \circ \Phi_\Omega \circ f_{\mathrm{F}} \circ g_\rho^{\mathrm{BNF}} = \Phi_{\Omega_\rho^{\mathrm{BNF}}} \circ f_{\mathrm{F}_\rho^{\mathrm{BNF}}}$$

$$\textbf{(6.154)} \qquad \Omega_\rho^{\mathrm{BNF}}(r) - \mathrm{BNF}(f)(r) = \mathrm{O}^{(1/\rho)^{1-\beta}}(r), \ \textit{in } \mathbf{R}[[r]]$$

$$\|\Omega_\rho^{\mathrm{BNF}}\|_{\mathrm{C}^3} \lesssim 1$$

$$\|g_\rho^{\mathrm{BNF}} - id\|_{\mathrm{C}^1} \leq \rho^{m-10}$$

$$\textbf{(6.155)} \qquad \|\mathrm{F}_\rho^{\mathrm{BNF}}\|_{\mathrm{W}_{h,\mathbf{D}(0,\rho^{b_\tau})}} \lesssim \exp(-(1/\rho)^{1-\beta}).$$

If $\Omega \in \mathcal{TC}(\mathrm{A},\mathrm{B})$ then $\Omega_\rho^{\mathrm{BNF}} \in \mathcal{TC}(2\mathrm{A}, 2\mathrm{B})$.

Proof. — See the Appendix, Section F.3. □

Remark **6.6.** — Inequality (6.155) can also be written

$$\|\mathrm{F}_{\rho^{1/\beta_\tau}}^{\mathrm{BNF}}\|_{\mathrm{W}_{h,\mathbf{D}(0,\rho)}} \lesssim \exp(-(1/\rho)^{(1-\beta)/b_\tau}).$$

We note that Iooss and Lombardi (Theorem 1.4 of [24]) obtained, for a similar problem, a more precise estimate but with essentially the same exponent $1/b_\tau = 1/(1+\tau)$ (their estimate reads $\leq (\mathrm{cst})\rho^2 \exp(-(\mathrm{cst})/\rho^{1/(1+\tau)})$).

Remark **6.7.** — In the (CC)-case and when ω_0 is in $\mathrm{DC}(\kappa,\tau)$, one can prove the previous proposition (maybe not with the same value for the exponent b_τ) by using Proposition 6.4 and the fact that $q_n \leq q_{n+1} \leq \kappa^{-1} q_n^\tau$.

6.4. *Consequence of the convergence of the BNF.*

Lemma **6.6.** — *Assume that* $\mathrm{BNF}(f)$ *coincides as a formal power series with a holomorphic function $\Xi \in \mathcal{O}(\mathbf{D}(0,\bar{\rho}))$ and, for $0 < \rho \leq \bar{\rho}$, let $\Omega \in \mathcal{O}(\mathbf{D}(0,\rho))$ be such that*

$$\textbf{(6.156)} \qquad \begin{cases} \Omega(r) - \mathrm{BNF}(f)(r) = \mathrm{O}^{\mathrm{N}+1}(r) & \textit{in } \mathbf{R}[[r]] \\ \|\Omega\|_{\mathbf{D}(0,\rho)} \leq 1. \end{cases}$$

Then

$$\|\Omega-\Xi\|_{\mathbf{D}(0,e^{-1}\rho)}\lesssim \exp(-\mathrm{N}).$$

Proof. — Let $\Xi(z)=\sum_{k=0}^{\infty}\xi_k z^k$, $\Omega(z)=\sum_{k=0}^{\infty}b_k z^k$, $\Xi_{\mathrm{N}}=\sum_{k=0}^{\mathrm{N}}\xi_k z^k$ and $\Omega_{\mathrm{N}}=\sum_{k=0}^{\mathrm{N}}b_k z^k$. We have from (6.156) and the fact that $\Xi=\mathrm{BNF}(f)$ in $\mathbf{R}[[r]]$

$$\Xi_{\mathrm{N}}=\Omega_{\mathrm{N}}. \tag{6.157}$$

On the other hand, we observe that if $g: z\mapsto \sum_{k\in\mathbf{N}}g_k z^k$ is in $\mathcal{O}(\mathbf{D}(0,\rho))$ one has by Cauchy's estimates $|g_k|\rho^k\le\|g\|_{\mathbf{D}(0,\rho)}$, hence for $|z|<e^{-1}\rho$

$$\begin{aligned}\Big|\sum_{k\ge\mathrm{N}+1}g_k z^k\Big| &\le \sum_{k\ge\mathrm{N}+1}\|g\|_{\mathbf{D}(0,\rho)}(z/\rho)^k\\ &\le 2e^{-\mathrm{N}}\|g\|_{\mathbf{D}(0,\rho)}.\end{aligned}$$

As a consequence,

$$\|\Xi-\Xi_{\mathrm{N}}\|_{\mathbf{D}(0,e^{-1}\rho)}\lesssim e^{-\mathrm{N}}\|\Xi\|_{\mathbf{D}(0,\overline{\rho})},\qquad \|\Omega-\Omega_{\mathrm{N}}\|_{\mathbf{D}(0,e^{-1}\rho)}\lesssim e^{-\mathrm{N}}\|\Omega\|_{\mathbf{D}(0,\overline{\rho})}.$$

We conclude using (6.157). □

To summarize,

Corollary **6.7.** — *If* $\mathrm{BNF}(\Phi_\Omega\circ f_{\mathrm{F}})$ *converges and coincide on* $\mathbf{D}(0,\overline{\rho})$ *with* $\Xi\in\mathcal{O}(\mathbf{D}(0,\overline{\rho}))$*, then for any* $\beta>0$ *and* $\rho\ll_\beta 1$ *one has:*

– *If* ω_0 *is* τ*-Diophantine ((AA) or (CC)-case)*

$$\|\Omega_\rho^{\mathrm{BNF}}-\Xi\|_{\mathbf{D}(0,\rho^{b_\tau})}\lesssim \exp(-(1/\rho)^{1-\beta}).$$

– *In the (CC) case for any* ω_0 *irrational*

$$\|\Omega_{q_{n+1}^{-1}}^{\mathrm{BNF}}-\Xi\|_{\mathbf{D}(0,q_{n+1}^{-6})}\lesssim \exp(-q_{n+1}^{1-\beta}).$$

7. KAM Normal Forms

We present now, in the unified (AA)-(CC) framework, the KAM scheme that is central in all this paper. This will be used in Sections 10 and 11 to construct the adapted Normal Forms and in Section 12 to get estimates on the Lebesgue measure of the set of KAM circles. For the sake of clarity we break down our main result into three propositions: Propositions 7.1, 7.2, 7.4.

As usual we denote in the (AA)-case $\mathrm{M}=\mathbf{T}_\infty\times\mathbf{C}$, $\mathrm{M}_{\mathbf{R}}=\mathbf{T}\times\mathbf{R}$, $\mathrm{O}=\mathbf{T}\times\{0\}$ and in the (CC)-case $\mathrm{M}=\mathbf{C}\times\mathbf{C}$ and $\mathrm{M}_{\mathbf{R}}=\mathrm{M}\cap\{r\in\mathbf{R}\}$, $\mathrm{O}=\{(0,0)\}$.

7.1. *The KAM statement.* — Let $0 < \overline{\rho} < h/2 < 1/2$, $A \geq 1$, $B \geq 1$ and $\Omega \in \widetilde{\mathcal{O}}_\sigma(e^h\mathbf{D}(0,\overline{\rho}))$ satisfying the following twist condition (see (2.59)):

$$\forall\, r \in \mathbf{R},\ A^{-1} \leq (1/2\pi)\partial^2\Omega(r) \leq A, \quad \text{and}\ \|(1/2\pi)D^3\Omega\|_{\mathbf{C}} \leq B. \tag{7.158}$$

Let $\omega(r) = (2\pi)^{-1}\partial\Omega(r)$. The image of $\mathbf{D}(0, e^h\overline{\rho})$ by ω is contained in a disk $\mathbf{D}(\omega(0), 3A\overline{\rho})$. We can assume without loss of generality that $\omega(0) \in [-1/2, 1/2]$ and consequently, if $\overline{\rho}$ is small enough we can assume

$$\omega(\mathbf{D}(0, e^h\rho) \cap \mathbf{R}) \subset [-3/4, 3/4]. \tag{7.159}$$

Let $\overline{C}$, $\overline{a}_0$ be the constants of Proposition 5.5. We introduce

$$\overline{a}_2 = 2(\overline{a}_0 + 2) + 10 \tag{7.160}$$

and assume that $F \in \mathcal{O}_\sigma(e^h W_{h,\mathbf{D}(0,\overline{\rho})})$ satisfies

$$\|F\|_{e^h W_{h,\mathbf{D}(0,\overline{\rho})}} \leq \overline{\rho}^{\,\overline{a}_2}. \tag{7.161}$$

By Cauchy's inequality (2.53) one has

$$\overline{\varepsilon} := \max_{0 \leq j \leq 3} \|D^j F\|_{W_{h,\mathbf{D}(0,\overline{\rho})}} \leq \overline{\rho}^{\,2(\overline{a}_0+2)+1}. \tag{7.162}$$

Associated to this $\overline{\varepsilon} > 0$ there exists a unique $N > 0$ such that

$$-\ln\overline{\varepsilon} = N/(\ln N)^2.$$

We then define for $n \geq 1$ the following sequences that depend on $\overline{\varepsilon} = \overline{\varepsilon}_1$, h and $\overline{\rho} > 0$:

$$\begin{cases} N_n = (4/3)^{n-1}N \\ \overline{\varepsilon}_n = e^{-hN_n/(\ln N_n)^2} \\ K_n^{-1} = \overline{\varepsilon}_n^{\frac{1}{2(\overline{a}_0+2)}}, \qquad (\overline{a}_0 \geq 5) \\ \delta_n = 2(\ln N_n)^{-2}h \\ \rho_n = \overline{\rho}\exp(-\sum_{j=1}^{n-1}\delta_j), \qquad h_n = h - (1/2)\sum_{j=1}^{n-1}\delta_j > h/2. \end{cases} \tag{7.163}$$

If $\overline{\rho}$ is small enough, for all $n \geq 1$ one has

$$\rho_n \geq e^{-1/20}\overline{\rho}, \qquad h_n \geq e^{-1/20}h$$

and (*cf.* (7.162)),

$$\rho_n/2 > 2K_n^{-1} \tag{7.164}$$

$$(\delta_n(2K_n^{-1}))^{-\overline{a}_0}K_n\overline{\varepsilon}_n < \overline{C}^{-1} \tag{7.165}$$

($\overline{C}$ is the constant of Proposition 5.5).

Proposition **7.1.** — *Assume that* Ω *and* F *are as above and that* $\overline{\rho} \ll_{A,B} 1$. *Then, with the notations (7.163) the following holds: for* $n \geq 1$ *there exist a decreasing (for the inclusion) sequence of holed domains* $(U_n)_{n\geq 1}$, *functions* $\Omega_n \in \widetilde{\mathcal{O}}_\sigma(U_n)$, $F_n \in \mathcal{O}_\sigma(\underbrace{W_{h_n,U_n}})$ *with* $U_1 = \mathbf{D}(0,\overline{\rho})$, $\Omega_1 = \Omega$, $F_1 = F$ *and, for* $n \geq 2$, $1 \leq m < n$, *diffeomorphisms* $g_{m,n} \in \mathrm{Symp}_{ex.,\sigma}(W_{h_n,U_n})$, *such that:*

$$\Omega_n \text{ satisfies a } (2A, 2B) - \text{twist condition, (cf. (2.59))} \tag{7.166}$$

$$g_{m,n}(W_{h_n,U_n}) \subset W_{h_m,U_m} \tag{7.167}$$

$$\text{on } W_{h_n,U_n}, \quad g_{m,n}^{-1} \circ \Phi_{\Omega_m} \circ f_{F_m} \circ g_{m,n} = \Phi_{\Omega_n} \circ f_{F_n} \tag{7.168}$$

$$\|g_{m,n} - id\|_{C^1} \leq \overline{\varepsilon}_m^{1/2}, \tag{7.169}$$

$$\max_{0\leq j\leq 3} \|D^j F_n\|_{W_{h_n,U_n}} \leq \overline{\varepsilon}_n. \tag{7.170}$$

Proof. — We construct inductively for $n \geq 2$ sequences U_n, F_n, Ω_n, $g_{m,n}$ satisfying the conclusion of the proposition with the additional requirements

Requirement 1: For $n \geq 2$, U_n is of the form

$$U_n = \mathbf{D}(0,\rho_n) \setminus \bigcup_{i\in I_n} \mathbf{D}(c_i,\kappa_i), \qquad c_i \in \mathbf{R}, \qquad \#I_n \leq 2N_{n-1}^2 \tag{7.171}$$

$$K_{n-1}^{-1} \leq \kappa_i \leq K_1^{-1} e^{\sum_{l=1}^{n-1}\delta_l}, \qquad \Big(\sum_{i\in I_n} \kappa_i^2\Big)^{1/2} \leq \sqrt{2} e^{\sum_{l=1}^{n-1}\delta_l} \sum_{l=1}^{n-1} N_l K_l^{-1}. \tag{7.172}$$

Requirement 2: For $n \geq 2$, $\Omega_n \in \widetilde{\mathcal{O}}_\sigma(U_n)$ satisfies an (A_n, B_n)-twist condition with (see (2.52) for the notation $\underline{a}(U_n)$)

$$1 \leq A_n \leq 2A - K_n^{-1}, \qquad 1 \leq B_n \leq 2B - K_n^{-1} \tag{7.173}$$

$$8 \max(\overline{\rho}, \underline{a}(U_n)) \times A_n \times B_n < 1. \tag{7.174}$$

$$\|\Omega_n - \Omega\|_{C^3(\mathbf{D}(0,\overline{\rho}))} \leq \sum_{l=1}^{n-1} \overline{\varepsilon}_l^{1/2} \leq 2\overline{\varepsilon}_1^{1/2} \tag{7.175}$$

$$\text{and} \quad \forall\, m < n, \ \|g_{m,n} - id\|_{C^1} \leq C \sum_{l=m}^{n-1} \varepsilon_l \lesssim \overline{\varepsilon}_m^{1/2} \qquad \text{(C from (2.43))}. \tag{7.176}$$

For some $n \geq 1$, assume the existence of U_n, F_n, Ω_n and the validity of conditions (7.171), (7.172), (7.173), (7.174), (7.175) (if $n \geq 2$) and define $\omega_n = (1/2\pi)\Omega_n$. Since (7.174) is satisfied we can apply Proposition 2.7 (with $A = A_n$, $B = B_n$, $3A_n\nu = K_n^{-1}$, $\beta = l/k$): for each $(k,l) \in \mathbf{Z}^2$, $0 < k < N_n$, such that $\mathbf{D}(l/k, (3A_nK_n)^{-1}) \cap \omega_n(U_n) \neq \emptyset$, there exists

$c^{(n)}_{l/k}\in\mathbf{R}$ such that

$$(\mathbf{7.177})\qquad \begin{cases}\omega_n(c^{(n)}_{l/k})=l/k\\ \forall\, r\in\mathbf{C}\setminus\overline{\mathbf{D}}(c^{(n)}_{l/k},\mathrm{K}_n^{-1}),\ |\omega_n(r)-(l/k)|\geq(3\mathrm{A}_n\mathrm{K}_n)^{-1}.\end{cases}$$

We denote

$$\begin{aligned}\mathrm{E}_n=\{(k,l)\in\mathbf{Z}^2,\ 0<k<\mathrm{N}_n,\ 0\leq|l|\leq\mathrm{N}_n,\\ \mathbf{D}(l/k,(3\mathrm{A}_n\mathrm{K}_n)^{-1})\cap\omega_n(\mathrm{U}_n)\neq\emptyset\}\end{aligned}$$

and we see that

$$(\mathbf{7.178})\qquad \#\mathrm{E}_n\leq 2\mathrm{N}_n^2.$$

Note that from (7.175) and (7.159) we have $|l/k|\leq 1$. Hence, if we define

$$(\mathbf{7.179})\qquad \mathrm{V}_n=\mathrm{U}_n\setminus\bigcup_{(k,l)\in\mathrm{E}_n}\mathbf{D}(c^{(n)}_{l/k},\mathrm{K}_n^{-1})$$

we have for any $r\in\mathrm{V}_n$ (*cf.* (7.173))

$$\forall\,(k,l)\in\mathbf{N}^*\times\mathbf{Z},\ 1\leq k<\mathrm{N}_n\implies\left|k\frac{1}{2\pi}\partial\Omega_n(r)-l\right|\geq(6\mathrm{A}\mathrm{K}_n)^{-1};$$

hence the non-resonance condition (5.126) (with $\tau=0$, $\mathrm{K}=6\mathrm{A}\mathrm{K}_n$, $\mathrm{N}=\mathrm{N}_n$) is satisfied. On the other hand (7.171)–(7.172) ($n\geq 2$) and (7.164) ($n=1$) show using (7.179) that (recall $\rho_n<\overline{\rho}<h/2$)

$$(\mathbf{7.180})\qquad \underline{\mathrm{d}}(\mathrm{W}_{h_n,\mathrm{V}_n})=\underline{\mathrm{d}}(\mathrm{V}_n)=\min(\underline{\mathrm{d}}(\mathrm{U}_n),\mathrm{K}_n^{-1})=\mathrm{K}_n^{-1}$$

and (7.165) and (7.170) show that

$$(\mathbf{7.181})\qquad (\delta_n\underline{\mathrm{d}}(\mathrm{W}_{h_n,\mathrm{V}_n}))^{-\overline{a}_0}\mathrm{K}_n\|\mathrm{F}_n\|_{h_n,\mathrm{U}_n}<\overline{\mathrm{C}}^{-1}.$$

We can thus apply Proposition 5.5 (with $\tau=0$, $\mathrm{K}=6\mathrm{A}\mathrm{K}_n$, $\delta=\delta_n$, $\mathrm{N}=\mathrm{N}_n$) on V_n: if one defines

$$(\mathbf{7.182})\qquad \mathrm{U}_{n+1}=e^{-\delta_n}\mathrm{V}_n=e^{-\delta_n}\mathrm{U}_n\setminus\bigcup_{(k,l)\in\mathrm{E}_n}\mathbf{D}(c^{(n)}_{l/k},e^{\delta_n}\mathrm{K}_n^{-1})$$

there exist $\mathrm{Y}_n\in\mathcal{O}(e^{-\delta_n/2}\mathrm{W}^{\Omega_n}_{h_n,\mathrm{V}_n})$, $\mathrm{F}_{n+1}\in\mathcal{O}_\sigma(\mathrm{W}_{h_{n+1},\mathrm{U}_{n+1}})$, $\Omega_{n+1}\in\mathcal{O}_\sigma(\mathrm{U}_{n+1})$ such that ($\overline{\rho}$ small enough)

$$(\mathbf{7.183})\qquad \|\mathrm{Y}_n\|_{e^{-\delta_n/2}\mathrm{W}_{h_n,\mathrm{V}_n}}\lesssim\mathrm{K}_n\delta_n^{-1}\|\mathrm{F}_n\|_{\mathrm{W}_{h_n,\mathrm{U}_n}}$$

$$(\mathbf{7.184})\qquad \mathrm{W}_{h_{n+1},\mathrm{U}_{n+1}},\qquad f_{\mathrm{Y}_n}\circ\Phi_{\Omega_n}\circ f_{\mathrm{F}_n}\circ f_{\mathrm{Y}_n}^{-1}=\Phi_{\tilde{\Omega}_{n+1}}\circ f_{\mathrm{F}_{n+1}}$$

$$\widetilde{\Omega}_{n+1} = \Omega_n + \mathrm{M}(\mathrm{F}_n) \tag{7.185}$$

$$\max_{0\le j\le 3} \|\mathrm{D}^j \mathrm{F}_{n+1}\|_{\mathrm{W}_{h_{n+1},\mathrm{U}_{n+1}}} \lesssim_{\mathrm{A}} \mathrm{K}_n(\delta_n \mathrm{K}_n^{-1})^{-\overline{a}_0}(\|\mathrm{F}_n\|^2_{\mathrm{W}_{h_n,\mathrm{U}_n}} + e^{-\delta_n \mathrm{N}_n/2}\|\mathrm{F}_n\|_{\mathrm{W}_{h_n,\mathrm{U}_n}}). \tag{7.186}$$

Let us show that the Requirements 1 (7.171)–(7.172) are satisfied for $n+1$. From (7.182) and (7.171) we see that

$$\mathrm{U}_{n+1} = \mathbf{D}(0, \rho_{n+1}) \setminus \bigcup_{i\in \mathrm{I}_{n+1}} \mathbf{D}(c_i, \kappa_i)$$

where $\mathrm{I}_{n+1} \le \mathrm{I}_n + 2\mathrm{N}_n^2$ (*cf.* (7.178)) and for all $i \in \mathrm{I}_{n+1}$, $\min(e^{\delta_n}\mathrm{K}_{n-1}^{-1}, e^{\delta_n}\mathrm{K}_n^{-1}) \le \kappa_i \le \mathrm{K}_1^{-1} e^{\sum_{l=1}^n \delta_l}$. Similarly, $(\sum_{i\in \mathrm{I}_{n+1}} \kappa_i^2)^{1/2} \le e^{\delta_n}((\sum_{i\in \mathrm{I}_n}\kappa_i^2)^{1/2} + (2\mathrm{N}_n^2\mathrm{K}_n^{-2})^{1/2})$. In other words, (7.171)–(7.172) are satisfied for $n+1$.

Let us now prove that the Requirements 2, (7.173), (7.174) (7.175) are satisfied for $n+1$ and in particular that $\widetilde{\Omega}_{n+1}$ has a nice Whitney extension $\Omega_{n+1} := \widetilde{\Omega}_{n+1}^{\mathrm{W}h}$. We first apply Lemma 2.2 to get a C^3, σ-symmetric extension $\mathrm{M}(\mathrm{F}_n)^{\mathrm{W}h} : \mathbf{C} \to \mathbf{C}$ for $(\mathrm{M}(\mathrm{F}_n), \mathrm{U}_n)$ such that

$$\sup_{0\le j\le 3} \|\mathrm{D}^j \mathrm{M}(\mathrm{F}_n)^{\mathrm{W}h}\|_{\mathbf{C}} \lesssim (1+\#\mathrm{J}_{\mathrm{U}_n})^3(\delta_n \underline{\mathrm{d}}(\mathrm{U}_n))^{-6} \max_{0\le j\le 3} \|\mathrm{D}^j \mathrm{M}(\mathrm{F}_n)\|_{e^{-\delta_n/10}\mathrm{U}_n}.$$

In particular, using Cauchy's inequalities, (7.171), (7.172), (7.163), (5.104) one gets

$$\begin{aligned}\sup_{0\le j\le 3} \|\mathrm{D}^j \mathrm{M}(\mathrm{F}_n)^{\mathrm{W}h}\|_{\mathbf{C}} &\lesssim \mathrm{N}_{n-1}^6(\delta_n \mathrm{K}_{n-1}^{-1})^{-6}\delta_n^{-3}\|\mathrm{M}(\mathrm{F}_n)\|_{\mathrm{U}_n}\\ &\lesssim \mathrm{K}_{n-1}^7 \overline{\varepsilon}_n \le \overline{\varepsilon}_n^{1/2}.\end{aligned} \tag{7.187}$$

From (7.185) we see that if we define the σ-symmetric function

$$\Omega_{n+1} := \Omega_n + \mathrm{M}(\mathrm{F}_n)^{\mathrm{W}h} \tag{7.188}$$

one has

$$\Omega_{n+1}\ \Big|_{\mathrm{U}_{n+1}} = \widetilde{\Omega}_{n+1}$$

and $(7.173)_{n+1}$, are satisfied (since $-\mathrm{K}_n^{-1} + \overline{\varepsilon}_n^{1/3} < -\mathrm{K}_{n+1}^{-1}$). To see that $(7.174)_{n+1}$ holds we use the fact that since the second inequality in (7.172) is true for $n+1$ (as already checked) one has $\underline{\mathrm{a}}(\mathrm{U}_{n+1}) \le (\sum_{i\in \mathrm{I}_{n+1}} \kappa_i^2)^{1/2} \le 2\sum_{l=1}^n \mathrm{N}_l \mathrm{K}_l^{-1} \le \mathrm{K}_1^{-1/2}$. If $\overline{\rho}$ is small enough we see that (7.162), (7.163) and $(7.173)_{n+1}$ ensure the validity of $(7.174)_{n+1}$.

Finally let us check $(7.176)_{n+1}$. From Lemma 2.2 we see that $(\mathrm{Y}_n, e^{-(3/4)\delta_n}\mathrm{W}^{\Omega_n}_{h_n,\mathrm{V}_n})$ has a C^3 σ-symmetric Whitney extension $\mathrm{Y}_n^{\mathrm{W}h}$ such that

$$\|\mathrm{Y}_n^{\mathrm{W}h}\|_{\mathrm{C}^3} \lesssim (1+\#\mathrm{J}_{\mathrm{V}_n})^3(\delta_n \underline{\mathrm{d}}(\mathrm{U}_n))^{-6} \max_{0\le j\le 2} \|\mathrm{D}^j \mathrm{Y}_n\|_{e^{-(2/3)\delta_n}\mathrm{V}_n}. \tag{7.189}$$

From (7.171), (7.179), (7.172) we see that $\#J_{V_n} \le 2N_n^2$, $\underline{d}(V_n) \ge K_n^{-1}$ hence using Cauchy's inequalities, (7.183), (7.163) and the fact that $\delta_n^{-1}, N_n \le K_n^{0+}$ and $K_n^7 \overline{\varepsilon}_n \le \overline{\varepsilon}_n^{(1/2)+}$, we get

$$\|Y_n^{Wh}\|_{C^3} \le \overline{\varepsilon}_n^{1/2}. \tag{7.190}$$

If we define $g_{n,n+1} = f_{Y_n^{Wh}}^{-1} \in \widetilde{\mathrm{Symp}}_{ex,\sigma}(W_{h_{n+1},U_{n+1}})$ and for $m \le n$, $g_{m,n+1} = g_{m,n} \circ g_{n,n+1}$ we have from (4.98) and (2.43)

$$\|g_{m,n+1} - id\|_{C^1} \le C(\|g_{m,n} - id\|_{C^1} + \|g_{n,n+1} - id\|_{C^1}) \le C \sum_{l=m}^{n} \overline{\varepsilon}_l \lesssim \overline{\varepsilon}_m$$

which is $(7.176)_{n+1}$ and implies $(7.169)_{n+1}$.

Note that $f_{Y_n^{Wh}}^{-1} = f_{Y_n}^{-1}$ on $W_{h_{n+1},U_{n+1}}$ and (7.184) shows that $(7.168)_{n+1}$ and $(7.167)_{n+1}$ are satisfied.

We now check that (7.170) holds for $n+1$; from (7.186) it is enough to verify

$$K_n^{\overline{a}_0+2}(\overline{\varepsilon}_n^2 + e^{-\delta_n N_n/2}\overline{\varepsilon}_n) < \overline{\varepsilon}_{n+1} \tag{7.191}$$

or equivalently since $e^{-\delta_n N_n/2} = \overline{\varepsilon}_n$, $K_n = \overline{\varepsilon}_n^{-\frac{1}{2(\overline{a}_0+2)}}$,

$$2\overline{\varepsilon}_n^{2-(1/2)} \le \overline{\varepsilon}_{n+1}$$

which is clearly satisfied since $3/2 > 4/3$, *cf.* (7.163). □

7.2. *Localization of the holes.* — We can localize the holes of the domains U_n:

Proposition **7.2** *(Localization of the holes).* — *For each $1 \le m < n$, one has*

$$\|\partial\Omega_n - \partial\Omega_m\|_{C^2} \lesssim \overline{\varepsilon}_m^{1/2} \tag{7.192}$$

and for some sets $E_i \subset \{(k,l) \in \mathbf{Z}^2,\ 0 < k < N_i,\ 0 \le |l| \le N_i\}$ $(1 \le i \le n-1)$ one can write U_n as

$$\mathbf{D}(0,\rho_n) \setminus \bigcup_{i=1}^{n-1} \bigcup_{(k,l)\in E_i} \mathbf{D}(c_{l/k}^{(i)}, s_{i,n-1}K_i^{-1}), \qquad s_{i,n-1} = e^{\sum_{l=i}^{n-1}\delta_l} \in [1,2] \tag{7.193}$$

where $\rho_n \ge e^{-1/5}\overline{\rho}$ and $c_{l/k}^{(i)}$ is on the real axis and is the unique solution of the equation $\omega_i(c_{l/k}^{(i)}) := (2\pi)^{-1}\partial\Omega_i(c_{l/k}^{(i)}) = l/k$.

Proof. — Inequality (7.192) is consequence of (7.188), (7.187). The expression (7.193) comes from (7.182). □

We now give a more detailed description of the structure of $\mathcal{D}(U_n)$, the set of holes of the domains U_n appearing in Proposition 7.2, *cf.* (7.193).

Lemma **7.3.** — *With the notations of Propositions 7.1–7.2:*

(1) For any $n_1 \le n_2$, $(k_j, l_j) \in \mathrm{E}_{n_j}, j = 1, 2$, one has

$$
\begin{cases} \text{if } l_1/k_1 = l_2/k_2 \text{ then } |c^{(n_1)}_{l_1/k_1} - c^{(n_2)}_{l_2/k_2}| \lesssim \overline{\varepsilon}^{1/2}_{n_1} \\ \text{if } l_1/k_1 \neq l_2/k_2 \text{ then } |c^{(n_1)}_{l_1/k_1} - c^{(n_2)}_{l_2/k_2}| \gtrsim \mathrm{N}^{-2}_{n_2}. \end{cases} \tag{7.194}
$$

(2) Let $n_1, n_2 \in \mathbf{N}$, $n_1 \le n_2$ and $0 < \kappa_2 < \kappa_1$ be such that

$$\kappa_1 + \kappa_2 \ll \mathrm{N}^{-2}_{n_2}, \qquad \overline{\varepsilon}^{1/2}_{n_1} \ll \kappa_1 - \kappa_2.$$

Then, two disks $\mathbf{D}(c^{(n_j)}_{l_j/k_j}, \kappa_j)$, $(k_j, l_j) \in \mathrm{E}_{n_j}$, $j = 1, 2$, are either disjoint or $l_1/k_1 = l_2/k_2$ and $\mathbf{D}(c^{(n_2)}_{l_2/k_2}, \kappa_2) \subset \mathbf{D}(c^{(n_1)}_{l_1/k_1}, \kappa_1)$.

Proof. — Item (1) is due to (7.192) and the fact that if $l_1/k_1 \neq l_2/k_2$

$$|(l_1/k_1) - (l_2/k_2)| \ge 1/(k_1k_2) \ge \mathrm{N}^{-2}_{n_2}.$$

Item (2) is a consequence of Item (1). Indeed, if $l_1/k_1 \neq l_2/k_2$ then since $\kappa_1 + \kappa_2 \ll \mathrm{N}^{-2}_{n_2}$ and $|c^{(n_1)}_{l_1/k_1} - c^{(n_2)}_{l_2/k_2}| \gtrsim \mathrm{N}^{-2}_{n_2}$ (we assume $n_1 \le n_2$), the disks $\mathbf{D}(c^{(n_1)}_{l_1/k_1}, \kappa_1)$ and $\mathbf{D}(c^{(n_2)}_{l_2/k_2}, \kappa_2)$ must have an empty intersection. On the other hand, if $l_1/k_1 = l_2/k_2$, then because of the fact that $|c^{(n_1)}_{l_1/k_1} - c^{(n_2)}_{l_2/k_2}| \lesssim \overline{\varepsilon}^{1/2}_{n_1}$, the disk $\mathbf{D}(c^{(n_1)}_{l_1/k_1}, \kappa_1)$ contains $\mathbf{D}(c^{(n_2)}_{l_2/k_2}, \kappa_2)$ because $\overline{\varepsilon}^{1/2}_{n_1} + \kappa_2 < \kappa_1$. □

7.3. *Whitney conjugation to an integrable model.* — By applying Lemma 2.2 one sees that $(\mathrm{F}_n, e^{-\delta_n}\mathrm{W}_{h_n,\mathrm{U}_n})$ and $(f_{\mathrm{F}_n}, e^{-\delta_n}\mathrm{W}_{h_n,\mathrm{U}_n})$ have C^3 real symmetric Whitney extensions $\mathrm{F}^{\mathrm{W}h}_n \in \widetilde{\mathcal{O}}_\sigma(e^{-\delta_n}\mathrm{W}_{h_n,\mathrm{U}_n}), f_{\mathrm{F}^{\mathrm{W}h}_n} \in \widetilde{\mathrm{Symp}}_{ex,\sigma}(\mathrm{W}_{h_n,\mathrm{U}_n})$ (the canonical map associated to $\mathrm{F}^{\mathrm{W}h}_n$) such that (see the discussion leading to (7.187) and inequality (4.98))

$$\|\mathrm{F}^{\mathrm{W}h}_n\|_{\mathrm{C}^3} \lesssim \overline{\varepsilon}^{1/2}_n, \qquad \|f_{\mathrm{F}^{\mathrm{W}h}_n} - id\|_{\mathrm{C}^1} \lesssim \overline{\varepsilon}^{1/3}_n.$$

We hence have

$$\text{on } e^{-\delta_n}\mathrm{W}_{h_n,\mathrm{U}_n}, \quad g^{-1}_{m,n} \circ \Phi_{\Omega_m} \circ f_{\mathrm{F}^{\mathrm{W}h}_m} \circ g_{m,n} = \Phi_{\Omega_n} \circ f_{\mathrm{F}^{\mathrm{W}h}_n}. \tag{7.195}$$

We show in the next Proposition that shrinking a little bit the domain of validity of the preceding formula one can impose that $g_{m,n}$ leaves invariant the origin $\mathrm{O} = \{r = 0\} \cap \mathrm{M}_{\mathbf{R}}$.

Lemma **7.4.** — *There exists $\widetilde{g}_{m,n} \in \widetilde{\mathrm{Symp}}_{ex,\sigma}(\mathrm{W}_{h_n/2,\mathrm{U}_n \setminus \mathbf{D}(0,\mathrm{K}^{-1}_m)})$ that coincides with $g_{m,n}$ on $\mathrm{W}_{h_n/2,\mathbf{C} \setminus \mathbf{D}(0,\mathrm{K}^{-1}_m)}$ and*

$$\widetilde{g}_{m,n}(\{r = 0\}) = \{r = 0\}, \qquad \|\widetilde{g}_{m,n} - id\|_{\mathrm{C}^1} \le \overline{\varepsilon}^{1/4}_m. \tag{7.196}$$

Proof. — Recall that $g_{m,n} = f^{-1}_{\mathrm{Y}^{\mathrm{W}h}_m} \circ \cdots \circ f^{-1}_{\mathrm{Y}^{\mathrm{W}h}_{n-1}}$ with $\mathrm{Y}^{\mathrm{W}h}_k \in \mathrm{C}^3 \cap \mathcal{O}_\sigma(e^{-(1/2)\delta_n}\mathrm{W}_{h_k,\mathrm{V}_k})$ satisfying (7.190). Let $\chi : \mathbf{R} \to [0, 1]$ be a smooth function with support in $[-1, 1]$

and equal to 1 on $[-1/2,1/2]$ and define the C^3 σ-symmetric function $\widetilde{Y}_k=(1-\chi((K_m r/2)^2))Y_k^{W^h}$. One has $\widetilde{Y}_k=Y_k^{W^h}$ on $W_{h,\mathbf{C}\setminus\mathbf{D}(0,K_m/2)}$ and $\|\widetilde{Y}_k\|_{C^3}\lesssim K_m^3\|Y^{W^h}\|_{C^2}\le\overline{\varepsilon}_m^{1/4}$ hence $f_{\widetilde{Y}_k}^{-1}$ coincide with $f_{Y_k^{W^h}}^{-1}$ on $W_{h,\mathbf{C}\setminus\mathbf{D}(0,K_m^{-1})}$ and $\|f_{\widetilde{Y}_k}^{-1}-id\|_{C^1}\lesssim\overline{\varepsilon}_k^{1/4}$. Since for $m\le k\le n-1$, $\widetilde{Y}_k$ is null on a neighborhood of $\{r=0\}$ the diffeomorphism $f_{\widetilde{Y}_k}^{-1}$ fixes $\{r=0\}$ and so does $\widetilde{g}_{m;n}$. The inequality in (7.196) follows from the fact that $\sum_{k=m}^{n-1}\overline{\varepsilon}_k^{1/4}\lesssim\overline{\varepsilon}_m^{1/4}$. □

Note that the sequence of diffeomorphisms $n\mapsto\widetilde{g}_{m,n}$ converges in C^1 to a σ-symmetric diffeomorphism $\widetilde{g}_{m,\infty}:\mathbf{C}\to\mathbf{C}$ fixing the origin and that satisfies $\|\widetilde{g}_{m,\infty}-id\|_{C^1}\lesssim\overline{\varepsilon}_m$. On the other hand, the sequence of diffeomorphisms $(f_{F_n^{W^h}})_n$ converges in C^1 to the identity and from (7.192) the sequence of functions $(\Omega_n)_n$, $\Omega_n\in\widetilde{\mathcal{O}}_\sigma(U_n)$ converges in C^2 to some σ-symmetric limit $\Omega_\infty\in C^2_\sigma(\mathbf{C})$; hence from (7.195)

$$\text{on }\bigcap_{n\ge m}e^{-\delta_n}W_{h_n/2,U_n\setminus\mathbf{D}(0,K_m^{-1})},\quad \widetilde{g}_{m,\infty}^{-1}\circ\Phi_{\Omega_m}\circ f_{F_m^{W^h}}\circ\widetilde{g}_{m,\infty}=\Phi_{\Omega_\infty}.$$

Recall the notations of Section 4.4 and let

$$L_m=\mathbf{R}\cap\bigcap_{n\ge m}e^{-\delta_n}(U_n\setminus\mathbf{D}(0,K_m^{-1})),$$

$$W_{L_m}=M_{\mathbf{R}}\cap\bigcap_{n\ge m}e^{-\delta_n}W_{h_n/2,U_n\setminus\mathbf{D}(0,K_m^{-1})}.$$

Proposition **7.5.** — *For any* $m\ge1$ *one has*

(7.197) $\quad$ *on* W_L, $\quad\widetilde{g}_{m,\infty}^{-1}\circ\Phi_{\Omega_m}\circ f_{F_m}\circ\widetilde{g}_{m,\infty}=\Phi_{\Omega_\infty}$

(7.198) $\quad\widetilde{g}_{m,\infty}(W_{L_m})\subset W_{\mathbf{R}\cap\overline{U}_m}$,

(7.199) $\quad\widetilde{g}_{m,n}(\{r=0\})=\{r=0\},\qquad\|\widetilde{g}_{m,n}-id\|_{C^1}\le\overline{\varepsilon}_m^{1/4}$

(7.200) $\quad\mathrm{Leb}_{\mathbf{R}}((\mathbf{R}\cap e^{-2\delta_m}U_m)\setminus L_m)\lesssim\overline{\varepsilon}_m^{\frac{1}{2(\overline{a}_0+3)}}.$

Proof. — Let us prove (7.198). Note that since $g_{m,n}$ and $\widetilde{g}_{m,n}$ coincide on $W_{h_n/2,\mathbf{C}\setminus\mathbf{D}(0,K_m^{-1})}$ one has from (7.167) $\widetilde{g}_{m,n}(e^{-\delta_n}W_{h_n,U_n\setminus\mathbf{D}(0,K_m^{-1})})\subset W_{h_m,U_m}$ hence since $\widetilde{g}_{m,n}$ is σ-symmetric, $\widetilde{g}_{m,n}(W_{L_m})\subset W_{\mathbf{R}\cap U_m}$ and $\widetilde{g}_{m,\infty}(L_m)\subset\overline{W_{\mathbf{R}\cap U_m}}=W_{\mathbf{R}\cap\overline{U}_m}$.

The conjugation relation (7.197) comes from the fact that $\Phi_{\Omega_m^{W^h}}\circ f_{F_m^{W^h}}$ coincides on $W_{\mathbf{R}\cap\overline{U}_m}$ with $\Phi_{\Omega_m}\circ f_{F_m}$.

For the proof of (7.200) we first observe that from the expression (7.193), for each $n>m$ the set $e^{-\sum_{l=m}^{n}\delta_l}U_m\setminus e^{-\delta_n}U_n$ is a union of at most $2N_n^2$ disks of radii $\le 2K_n^{-1}$ hence the Lebesgue measure of its intersection with $M_{\mathbf{R}}$ is $\le 4N_n^2K_n^{-1}$. In consequence, the Lebesgue measure of $\mathbf{R}\cap e^{-\sum_{l=m}^{\infty}\delta_l}U_m\setminus\bigcap_{n\ge m+1}e^{-\delta_n}U_n$ is $\lesssim\sum_{n=m+1}^{\infty}N_n^2K_n^{-1}\le\overline{\varepsilon}_m^{\frac{1}{2(\overline{a}_0+3)}}$

hence

$$\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}(e^{-2\delta_m}\mathrm{U}_m \smallsetminus \bigcap_{n\geq m} e^{-\delta_n}\mathrm{U}_n) \lesssim \overline{\varepsilon}_m^{\frac{1}{2(\overline{a}_0+3)}}$$

and since $\mathrm{L} \supset (\bigcap_{n\geq m} e^{-\delta_n}\mathrm{U}_n) \smallsetminus e^{\delta_m}\mathbf{D}(0, \mathrm{K}_m^{-1})$ we get that $\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}((\mathbf{R}\cap e^{-2\delta_m}\mathrm{U}_m) \smallsetminus \mathrm{L}_m)) \leq \overline{\varepsilon}_m^{\frac{1}{2(\overline{a}_0+3)}} + e^{\delta_m}\mathrm{K}_m^{-1}$; (7.200) follows from this inequality. □

Remark **7.1.** — If U is a holed domain, Propositions 7.1, 7.2, 7.5 as well as their proofs, extend without any change to the situation where $\mathrm{F} \in \mathcal{O}_\sigma(e^h \mathrm{W}_{h,\mathrm{U}})$ and $\Omega \in \widetilde{\mathcal{O}}_\sigma(e^h\mathrm{U})$ satisfies the twist condition (7.158)–(2.60) and if the following smallness assumption on F holds

$$\|\mathrm{F}\|_{e^h\mathrm{W}_{h,\mathrm{U}}} \leq \underline{\mathrm{d}}(\mathrm{W}_{h,\mathrm{U}})^{\overline{a}_2}. \tag{7.201}$$

8. Hamilton-Jacobi Normal Form and the Extension Property

Our aim in this section is to provide a useful approximate Normal Form (that we call the Hamilton-Jacobi Normal Form) in a neighborhood of a *q-resonant circle* $\{r = c\}$ (by which we mean that for some $(p, q) \in \mathbf{Z}\times\mathbf{N}^*$, $p\wedge q = 1$ $\omega(c) = \frac{p}{q}$).

Let $0 < \widehat{\rho} < h/2 < 1/20$, $c \in \mathbf{R}$, $(p, q) \in \mathbf{Z}\times\mathbf{N}^*$, $p \wedge q = 1$, $\Omega \in \widetilde{\mathcal{O}}_\sigma(\mathbf{D}(c, 6\widehat{\rho}))$, $\mathrm{F} \in \mathcal{O}_\sigma(\mathrm{W}_{h,\mathbf{D}(c,6\widehat{\rho})})$ such that

$$\forall\, r \in \mathbf{R},\ \mathrm{A}^{-1} \leq (2\pi)^{-1}\partial^2\Omega(r) \leq \mathrm{A}, \quad \text{and} \quad \|(2\pi)^{-1}\mathrm{D}^3\Omega\|_{\mathbf{C}} \leq \mathrm{B}. \tag{8.202}$$

$$\overline{\varepsilon} := \|\mathrm{F}\|_{\mathrm{W}_{h,\mathbf{D}(c,6\widehat{\rho})}} \leq \min((6\widehat{\rho})^{\overline{a}_3}, (10\mathrm{A})^{-8}), \tag{8.203}$$

$$\omega(c) := (2\pi)^{-1}\partial\Omega(c) = \frac{p}{q} \tag{8.204}$$

$$(6\widehat{\rho})^{1/8} < (\mathrm{A}q)^{-1} < h/10, \qquad 6\widehat{\rho} < |c|/4 \tag{8.205}$$

where $\overline{a}_3$ is the constant appearing in Proposition G.1 of Appendix G on Resonant Normal Forms.

The purpose of this section is to prove the following result:

Proposition **8.1** *(Hamilton-Jacobi Normal Form). — Let $\widehat{\mathrm{D}} = \mathrm{D}(c, \widehat{\rho})$. There exists a disk $\check{\mathrm{D}}$,*

$$\check{\mathrm{D}} := \mathbf{D}(\check{c}, \check{\rho}) \subset \widehat{\mathrm{D}} = \mathrm{D}(c, \widehat{\rho}), \quad \textit{with} \quad \check{\rho} \leq \overline{\varepsilon}^{1/33} \tag{8.206}$$

and

$$\Omega^{\mathrm{HJ}}_{\widehat{\mathrm{D}}} \in \widetilde{\mathcal{O}}_\sigma(\widehat{\mathrm{D}} \smallsetminus \check{\mathrm{D}})), \quad \mathrm{F}^{\mathrm{HJ}}_{\widehat{\mathrm{D}}} \in \mathcal{O}_\sigma(\mathrm{W}_{h/9,(\widehat{\mathrm{D}}\smallsetminus\check{\mathrm{D}})}), \quad g^{\mathrm{HJ}}_{\widehat{\mathrm{D}}} \in \widetilde{\mathrm{Symp}}_\sigma((\mathrm{W}_{h/9,(\widehat{\mathrm{D}}\smallsetminus\check{\mathrm{D}})})$$

such that

$$(\mathbf{8.207}) \qquad \Omega^{\mathrm{HJ}}_{\widehat{\mathrm{D}}} \textit{ satisfies a } (2\mathrm{A}, 2\mathrm{B}) - \textit{twist condition}$$

$$(\mathbf{8.208}) \qquad \mathrm{W}_{h/9,(\widehat{\mathrm{D}}\smallsetminus\check{\mathrm{D}})}, \qquad (g^{\mathrm{HJ}}_{\widehat{\mathrm{D}}})^{-1}\circ\Phi_\Omega\circ f_{\mathrm{F}}\circ g^{\mathrm{HJ}}_{\widehat{\mathrm{D}}} = \Phi_{\Omega^{\mathrm{HJ}}_{\widehat{\mathrm{D}}}}\circ f_{\mathrm{F}^{\mathrm{HJ}}_{\widehat{\mathrm{D}}}}$$

$$(\mathbf{8.209}) \qquad \|g^{\mathrm{HJ}}_{\widehat{\mathrm{D}}} - id\|_{\mathrm{C}^1} \lesssim q\overline{\varepsilon}^{1/8}$$

$$(\mathbf{8.210}) \qquad \|\mathrm{F}^{\mathrm{HJ}}_{\widehat{\mathrm{D}}}\|_{\mathrm{W}_{h/9,(\widehat{\mathrm{D}}\smallsetminus\check{\mathrm{D}})}} \lesssim \exp(-1/(6q\widehat{\rho})^{1/4})\overline{\varepsilon}.$$

Moreover, one has the following:

Extension property: $(\Omega^{\mathrm{HJ}}_{\mathrm{D}}, \widehat{\mathrm{D}}, \check{\mathrm{D}})$ *satisfies the following Extension Principle: If there exists a holomorphic function* $\Xi\in\mathcal{O}(\widehat{\mathrm{D}})$ *such that*

$$\|\Omega^{\mathrm{HJ}}_{\mathrm{D}} - \Xi\|_{(4/5)\widehat{\mathrm{D}}\smallsetminus(1/5)\widehat{\mathrm{D}}} \lesssim \nu$$

then $\check{\rho}\lesssim\nu^{1/200}$.

Remark **8.1.** — From Lemma K.1 and Remark K.1 of the Appendix, we just have to prove the Proposition in the (AA)-setting. This is the setting in which we shall work in all this section.

The proof of the first part of Proposition 8.1 is done in Section 8.7 and that of the second part (Extension Principle), based on Proposition 8.10, is done in Section 8.9.

From now on we define

$$\overline{\rho} = 6\widehat{\rho}.$$

8.1. *Putting the system into Resonant Normal Form.* — From Proposition G.1 of the Appendix on the existence of approximate q-Resonant Normal Form, we know that there exist $\overline{\Omega}\in\widetilde{\mathcal{O}}_\sigma(\mathbf{D}(c,e^{-1/q}\overline{\rho}))$, $g_{\mathrm{RNF}}\in\widetilde{\mathrm{Symp}}_{ex,\sigma}(e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})})$, $\overline{\mathrm{F}}^{res}, \mathrm{F}^{cor}\in\mathcal{O}_\sigma(e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})})$ such that $\overline{\mathrm{F}}^{res}$ is $2\pi/q$-periodic, $\mathcal{M}_0(\overline{\mathrm{F}}^{res})=0$, and

$$(\mathbf{8.211}) \qquad \begin{cases} e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}, \quad g^{-1}_{\mathrm{RNF}}\circ\Phi_\Omega\circ f_{\mathrm{F}}\circ g_{\mathrm{RNF}} = \Phi_{2\pi(p/q)r}\circ\Phi_{\overline{\Omega}}\circ f_{\overline{\mathrm{F}}^{res}}\circ f_{\mathrm{F}^{cor}} \\ \overline{\mathrm{F}}^{res} \text{ is } 2\pi/q - \text{periodic}, \qquad \mathcal{M}_0(\overline{\mathrm{F}}^{res}) = 0, \end{cases}$$

with

$$(\mathbf{8.212}) \qquad \begin{cases} \|\overline{\Omega} - (\Omega - 2\pi(p/q)r)\|_{\mathbf{D}(c,e^{-1/q}\overline{\rho})} \lesssim \|\mathrm{F}\|_{\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \\ \|\overline{\mathrm{F}}^{res}\|_{e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim \|\mathrm{F}\|_{\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \\ \|\mathrm{F}^{cor}\|_{e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim \exp(-\overline{\rho}^{-1/4})\|\mathrm{F}\|_{\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \\ \|g_{\mathrm{RNF}} - id\|_{\mathrm{C}^1} \le (q\overline{\rho}^{-1})^5\|\mathrm{F}\|_{h,\mathbf{D}(c,\overline{\rho})} \end{cases}$$

Inequalities (8.212) and the fact that Ω satisfies an (A, B)-twist condition on $\mathbf{D}(0,\overline{\rho})$ show that there exists a unique $\overline{c}\in\mathbf{R}$ such that

$$\partial\overline{\Omega}(\overline{c})=0,\qquad |\overline{c}-c|\lesssim\overline{\varepsilon}.$$

8.2. *Coverings.* — We denote $\mathbf{R}_h=\mathbf{R}+\sqrt{-1}[-h,h]$ and by j_q the q-covering

$$(\mathbf{8.213})\qquad \begin{aligned} j_q\ \ &(\mathbf{C}/(2\pi\mathbf{Z}))\times\mathbf{C}\to(\mathbf{C}/(2\pi/q)\mathbf{Z})\times\mathbf{C}\\ &(\theta+2\pi\mathbf{Z},r)\mapsto(\theta+(2\pi/q)\mathbf{Z},r).\end{aligned}$$

Since the function $\overline{\mathrm{F}}^{res}:(\theta,r):(\mathbf{R}_{h-2/q}/(2\pi)\mathbf{Z})\times\mathbf{D}(\overline{c},e^{-2/q}\overline{\rho})\to\mathbf{C}$ is invariant by $(\theta,r)\mapsto(\theta+2\pi/q,r)$ one can push it down to a function

$$\overline{\mathrm{F}}^{res}_{j_q}:(\mathbf{R}_{h-2/q}/(2\pi/q)\mathbf{Z})\times\mathbf{D}(\overline{c},e^{-2/q}\overline{\rho})\to\mathbf{C},\qquad \overline{\mathrm{F}}^{res}_{j_q}\circ j_q=\overline{\mathrm{F}}^{res}.$$

Let

$$(\mathbf{8.214})\qquad \begin{aligned}\Lambda_q\ :\ &(\mathbf{C}/(2\pi/q)\mathbf{Z})\times\mathbf{C}\to\mathbf{C}/(2\pi)\mathbf{Z}\times\mathbf{C}\\ &(\theta,r)\mapsto(q\theta,q(r-\overline{c}))\end{aligned}$$

and define $\widetilde{\mathrm{F}}^{res}:(\mathbf{R}_{qh-2}/(2\pi)\mathbf{Z})\times\mathbf{D}(0,e^{-2/q}q\overline{\rho})\to\mathbf{C}$ by

$$\widetilde{\mathrm{F}}^{res}=q^2\overline{\mathrm{F}}^{res}_{j_q}\circ\Lambda_q^{-1};$$

for all $(\widetilde{\theta},\widetilde{r})\in\mathbf{T}_{qh-2}\times\mathbf{D}(0,qe^{-2/q}\overline{\rho})$ and $(\theta,r)\in\mathbf{T}_{h-2/q}\times\mathbf{D}(\overline{c},e^{-2/q}\overline{\rho})$ such that $\widetilde{\theta}=q\theta,\widetilde{r}=q(r-\overline{c})$ one has

$$(\mathbf{8.215})\qquad \widetilde{\mathrm{F}}^{res}(\widetilde{\theta},\widetilde{r})=q^2\overline{\mathrm{F}}^{res}(\theta,\overline{c}+r).$$

Let $f_{\widetilde{\mathrm{F}}^{res}}$ be the (exact) symplectic mapping (for the symplectic form $d\widetilde{\theta}\wedge d\widetilde{r}$) defined by (4.87): if $(\widetilde{\varphi},\widetilde{\mathrm{R}})=\Lambda(\varphi,\mathrm{R})$, $(\widetilde{\theta},\widetilde{r})=\Lambda(\theta,r)$

$$(\widetilde{\varphi},\widetilde{\mathrm{R}})=f_{\widetilde{\mathrm{F}}^{res}}(\widetilde{\theta},\widetilde{r})\iff(\varphi,\mathrm{R})=f_{\overline{\mathrm{F}}^{res}_{j_q}}(\theta,r).$$

If we set

$$(\mathbf{8.216})\qquad \begin{aligned}\widetilde{\Omega}(r)&:=q^2\Big(\overline{\Omega}(\overline{c}+(r/q))-2\pi(p/q)(r/q)\Big)\\ &=(1/2)\partial^2\overline{\Omega}(\overline{c})r^2+\mathrm{O}(r^3)\\ &=\varpi r^2+r^3b(r)\end{aligned}$$

we have

$$(\mathbf{8.217})\qquad \Lambda_q\circ\Phi_{\overline{\Omega}}\circ f_{\overline{\mathrm{F}}^{res}_{j_q}}\circ\Lambda_q^{-1}=\Phi_{\widetilde{\Omega}}\circ f_{\widetilde{\mathrm{F}}^{res}}.$$

Note that since Ω satisfies an (A, B)-twist condition, one has from the first equation of (8.212) (to which one applies Cauchy's inequality), the estimate

$$\forall\, r\in \mathbf{D}(0,e^{-1/10}\overline{\rho}),\qquad \partial^2\widetilde{\Omega}(r)\asymp 1,\qquad \|\widetilde{\Omega}\|_{C^3(\mathbf{D}(0,e^{-1/10}q\overline{\rho}))}\lesssim 1. \tag{8.218}$$

8.3. *Approximation by a Hamiltonian flow.* — The following proposition says that up to some very good approximation $\Phi_{\widetilde{\Omega}}\circ f_{\widetilde{F}^{res}}$ can be seen as the time-1 map of a Hamiltonian vector field in the plane.

Proposition **8.2.** — *There exists* $\widetilde{F}^{vf},\widetilde{F}^{per}\in\mathcal{O}_\sigma(\mathbf{T}_{e^{-2/q}qh/2}\times\mathbf{D}(0,e^{-2/q}q\overline{\rho}/2))$*, such that on* $\mathbf{T}_{e^{-2/q}qh/2}\times\mathbf{D}(0,e^{-2/q}q\overline{\rho}/2)$ *one has*

$$\Phi_{\widetilde{\Omega}}\circ f_{\widetilde{F}^{res}}=\Phi_{\widetilde{\Omega}+\widetilde{F}^{per}}\circ f_{\widetilde{F}^{vf}} \tag{8.219}$$

$$\widetilde{F}^{per}=\widetilde{F}^{res}+O(\overline{\rho}^{\,1/4}\|\widetilde{F}^{res}\|_{\mathbf{T}_{e^{-2/q}qh}\times\mathbf{D}(0,e^{-2/q}q\overline{\rho})})=O(q^2\|F\|_{h,\mathbf{D}(0,\overline{\rho})}) \tag{8.220}$$

$$\|\widetilde{F}^{vf}\|_{e^{-2/q}qh/2,\mathbf{D}(0,e^{-2/q}q\overline{\rho}/2)}\lesssim \exp(-1/(q\overline{\rho})^{1/4})\|F\|_{h,\mathbf{D}(0,\overline{\rho})}. \tag{8.221}$$

Proof. — This is a consequence of (8.212), (8.215) and Proposition H.1 applied to $\Phi_{\widetilde{\Omega}}\circ f_{\widetilde{F}^{res}}$ (since by (8.212), (8.215), condition (H.545) is satisfied). □

Let $F(\theta,r)=\sum_{i=0}^{2}f_i(\theta)r^i+r^3\widetilde{f}(\theta,r)$ and define (*cf.* (8.216))

$$\widetilde{\Pi}(\theta,r)=\widetilde{\Omega}(r)+\widetilde{F}^{per}(\theta,r) \tag{8.222}$$

$$=:\varpi r^2+f_0(\theta)+f_1(\theta)r+f_2(\theta)r^2+r^3(b(r)+\widetilde{f}(\theta,r)) \tag{8.223}$$

$$=(\varpi+f_2(\theta))\left(r+\frac{1}{2}\frac{f_1(\theta)}{\varpi+f_2(\theta)}\right)^2-\frac{1}{4}\frac{f_1(\theta)^2}{\varpi+f_2(\theta)}+f_0(\theta)+r^3(b(r)+\widetilde{f}(\theta,r)) \tag{8.224}$$

where

$$\max_{\mathbf{T}_{e^{-2/q}qh/2}\times\mathbf{D}(0,e^{-2/q}q\overline{\rho}/2)}(|f_0|,|f_1|,|f_2|,|\widetilde{f}|)\lesssim (q\overline{\rho})^{-3}q^2\overline{\varepsilon}. \tag{8.225}$$

8.4. *From* $\widetilde{\Pi}$ *to* $\overline{\Pi}$. — We assume in the rest of this section that $\varpi>0$ and we set

$$\rho_q=q\overline{\rho}/3. \tag{8.226}$$

The next lemma provides a more convenient expression for the function, viewed as a Hamiltonian, $\widetilde{\Pi}=\widetilde{\Omega}+\widetilde{F}^{per}$ which was defined in (8.222).

Lemma **8.3.** — *There exists a (not exact) symplectic change of coordinates* $G \in \mathrm{Symp}^{\mathcal{O}}_{\sigma}(\mathbf{T}_{qh/3} \times \mathbf{D}(0, \rho_q))$ *of the form* $G(\theta, r) = (\theta, r - e_0(\theta))$ *and* $\overline{\Pi} \in \mathcal{O}(\mathbf{T}_{qh/3} \times \mathbf{D}(0, e^{-1/10}\rho_q))$ *such that*

$$\overline{\Pi}(\theta, r) := \widetilde{\Pi} \circ G^{-1}(\theta, r) = \varpi(\theta)(r^2 - e_1(\theta) + r^3 f(\theta, r)) \tag{8.227}$$

with $\varpi, e_0, e_1 \in \mathcal{O}_\sigma(\mathbf{T}_{qh/3}), f \in \mathcal{O}_\sigma(\mathbf{T}_{qh/3} \times \mathbf{D}(0, \rho_q))$,

$$\|\varpi(\cdot) - \varpi\|_{qh/3} \lesssim q\overline{\rho}^{-2}\overline{\varepsilon}, \quad \max(\|e_0\|_{qh/3}, \|e_1\|_{qh/3}), \lesssim q\overline{\rho}^{-1}\overline{\varepsilon}, \quad \|f\|_{qh/3,\rho_q} \lesssim 1. \tag{8.228}$$

Proof. — See Appendix L.1. □

Remark **8.2.** — The previous lemma and (8.224) show that

$$\varpi(\theta) = \varpi + f_2(\theta) + O(\rho_q^3 \overline{\varepsilon})$$

and

$$e_0(\theta) = -\frac{1}{2}\frac{f_1(\theta)}{\varpi + f_2(\theta)} + O(\rho_q^3\overline{\varepsilon}),$$
$$e_1(\theta) = -\frac{1}{4}\frac{f_1(\theta)^2}{(\varpi + f_2(\theta))^2} + \frac{f_0(\theta)}{\varpi + f_2(\theta)} + O(\rho_q^3\overline{\varepsilon}).$$

Remark **8.3.** — Since $\widetilde{\Pi}$ is defined up to an additive constant (this will not change the value of e_0), we can assume that

$$\int_{\mathbf{T}} \frac{\widetilde{\Pi}(\theta, e_0(\theta))}{\varpi(\theta)^{1/2}} \frac{d\theta}{2\pi} = 0$$

which is equivalent to the following condition that we will assume to hold from now on

$$\int_{\mathbf{T}} \varpi(\theta)^{1/2} e_1(\theta) \frac{d\theta}{2\pi} = 0. \tag{8.229}$$

8.5. *Hamilton-Jacobi Normal Form for* $\overline{\Pi}$. — The symplectic diffeomorphism $\Phi_{\overline{\Pi}}$ is the time-1 map of a Hamiltonian defined on the cylinder, and as such, it is integrable in the Hamilton-Jacobi sense: the level lines of the Hamiltonian foliate the cylinder and naturally provide invariant curves for the Hamiltonian flow. On some open sets[41] it is possible to conjugate $\Phi_{\overline{\Pi}}$ to a Hamiltonian depending only on the action variable: this is the Hamilton-Jacobi Normal Form; see Proposition 8.7. The purpose of this Subsection is to quantify this fact.

[41] These are cylindrical domains outside the "eyes" defined by separatrices (think of a pendulum).

Recall the expression for $\overline{\Pi}$

$$\overline{\Pi}(\theta, r) = \varpi(\theta)(r^2 - e_1(\theta) + r^3 f(\theta, r)).$$

Let $0 \le s \le h/3$. We denote

$$\varepsilon_1 := \|e_1\|_{C^0(\mathbf{T})} \lesssim q\overline{\rho}^{-1}\overline{\varepsilon}, \qquad \varepsilon_{1,s} = \varepsilon_1(s) = \|e_1\|_{qsh/3}, \tag{8.230}$$

and for $L \gg 1$ we introduce

$$\lambda_{0,L} := L\varepsilon_1^{1/2}, \qquad \lambda_{s,L} = \lambda(s, L) = L\varepsilon_{1,s}^{1/2}, \tag{8.231}$$

with the requirement

$$\lambda_{s,L} < q\overline{\rho}/6 = \rho_q/2 \quad \text{or equivalently} \quad 1 \ll L \lesssim q\overline{\rho}\varepsilon_{1,s}^{-1/2}. \tag{8.232}$$

We notice that $0 < \lambda_{s,L} \lesssim \rho_q$ and that from the Three Circles Theorem

$$\varepsilon_1(0) \le \varepsilon_1(s) \le \varepsilon_1(0)^{1-s}\varepsilon_1(1)^s \tag{8.233}$$

hence

$$\lambda_{0,L} = L\varepsilon_1^{1/2} \le \lambda_{s,L} \le L\varepsilon_1^{(1-s)/2}. \tag{8.234}$$

Notation **8.4.** — *For* $0 < a_1 < a_2$ *and* $z \in \mathbf{C}$ *we denote by* $\mathbf{A}(z; a_1, a_2)$ *the annulus centered at* z *with inner and outer radii of sizes respectively* a_1 *and* a_2*. When* $z = 0$ *we simply denote this annulus by* $\mathbf{A}(a_1, a_2)$*.*

Before giving the Hamilton-Jacobi Normal Form of $\overline{\Pi}$ we need two lemmas.

Lemma **8.5.** — *There exists a holomorphic function* g *defined on*

$$\mathrm{Dom}(g) := \bigcup_{0 \le s \le 1} (\mathbf{T}_{qsh/3} \times \mathbf{A}(\lambda_{s,L}, \rho_q))$$

such that for every $(\theta, z) \in \mathrm{Dom}(g)$ *one has*

$$\overline{\Pi}(\theta, g(\theta, z)) = z^2. \tag{8.235}$$

Moreover, there exists $\mathring{g} \in \mathcal{O}(\mathrm{Dom}(g))$ *such that on* $\mathrm{Dom}(g)$ *one has*

$$g(\theta, z) = \varpi(\theta)^{-1/2} z(1 + \mathring{g}(\theta, z)), \qquad \|\mathring{g}\|_{\mathrm{Dom(g)}} \lesssim L^{-2}. \tag{8.236}$$

Proof. — See the Appendix, Section L.3. □

Since $\mathbf{T}\times\mathbf{A}(\lambda_{0,\mathrm{L}},\rho_q)\subset \mathrm{Dom}(g)$ we can define the function $\Gamma\in\mathcal{O}(\mathbf{A}(\lambda_{0,\mathrm{L}},\rho_q))$ by $\Gamma:\mathbf{A}(\lambda_{0,\mathrm{L}},\rho_q)\to\mathbf{C}$

$$\Gamma(u)=(2\pi)^{-1}\int_0^{2\pi} g(\varphi,u)d\varphi. \tag{8.237}$$

Using (8.236) we see that Γ can be written

$$\Gamma(u)=\gamma u(1+\mathring{\Gamma}(u)),\qquad \gamma:=(2\pi)^{-1}\int_0^{2\pi}\varpi(\theta)^{-1/2}d\theta,$$
$$\|\mathring{\Gamma}\|_{\mathbf{A}(\lambda_{s,\mathrm{L}},\rho_q)}\lesssim \mathrm{L}^{-2}.$$

Lemma **8.6.** — *There exists a solution* $\mathrm{H}\in\mathcal{O}(\mathbf{A}(2\lambda_{s,\mathrm{L}},\rho_q/2))$ *of the equation*

$$\Gamma(\mathrm{H}(z))=z. \tag{8.238}$$

Moreover it can be written

$$\mathrm{H}(z)=\gamma^{-1}z(1+\mathring{\mathrm{H}}(z)),\qquad \|\mathring{\mathrm{H}}\|_{\mathbf{A}(2\lambda_{s,\mathrm{L}},(1/2)\rho_q)}\le \mathrm{L}^{-2}. \tag{8.239}$$

Proof. — See the Appendix, Section L.4. □

We now apply the preceding results with

$$s=1/2.$$

Proposition **8.7** *(Hamilton-Jacobi).* — *There exists an exact symplectic change of coordinates* $\overline{\mathrm{W}}\in\widetilde{\mathrm{Symp}}_{ex,\sigma}(\mathbf{T}_{qh/7}\times\mathbf{A}(3\varepsilon_1^{1/32},\rho_q/3))$ *such that*

$$\overline{\mathrm{W}}^{-1}\circ\Phi_{\overline{\Pi}}\circ\overline{\mathrm{W}}=\Phi_{\mathrm{H}^2} \tag{8.240}$$

$$\|\overline{\mathrm{W}}-id\|_{\mathrm{C}^1}\lesssim q\varepsilon_1^{1/4}. \tag{8.241}$$

Proof. — Let H be the function defined by the previous lemma (with $s=1/2$) and define for $z\in\mathbf{A}(2\lambda_{1/2,\mathrm{L}},\rho_q/2)$ and $\theta\in\mathrm{J}_{qh/6}:=[-4\pi,4\pi]+i[-qh/6,qh/6]$

$$\mathrm{S}(\theta,z)=\int_{[0,\theta]} g(\varphi,\mathrm{H}(z))d\varphi. \tag{8.242}$$

We notice that by Cauchy's Formula, (8.237) and (8.238)

$$\begin{aligned}\mathrm{S}(\theta+2\pi,z)-\mathrm{S}(\theta,z)&=\int_{[\theta,\theta+2\pi]} g(\varphi,\mathrm{H}(z))d\varphi\\ &=\int_0^{2\pi} g(\varphi,\mathrm{H}(z))d\varphi\end{aligned}$$

$$= 2\pi\Gamma(H(z))$$
$$= 2\pi z$$

hence

$$\Sigma : (\theta, z) \mapsto S(\theta, z) - \theta z$$

defines a holomorphic function on $\mathbf{T}_{qh/6} \times \mathbf{A}(2\lambda_{1/2,L}, \rho_q/2)$. Moreover, from (8.242), (8.236) and (8.239) one can write

$$\begin{aligned} S(\theta, z) &= \int_0^\theta g(\varphi, H(z))d\varphi \\ &= \int_0^\theta \varpi(\varphi)^{-1/2} H(z)(1 + \mathring{g}(\varphi, H(z)))d\varphi \\ &= \gamma\theta H(z) + \int_0^\theta \varpi(\varphi)^{-1/2} H(z)\mathring{g}(\varphi, H(z))d\varphi \\ &= \theta z(1 + \mathring{H}(z)) + \int_0^\theta \varpi(\varphi)^{-1/2} H(z)\mathring{g}(\varphi, H(z))d\varphi \end{aligned}$$

and we see that

$$\|\Sigma\|_{\mathbf{T}_{qh/6}\times\mathbf{A}(2\lambda_{1/2,L},\rho_q/2)} \lesssim L^{-2}(1 + qh/6).$$

Define $U_{L,\delta} = \mathbf{T}_{qh/6-\delta} \times \mathbf{A}(2L\varepsilon_1^{1/4} + \delta, \rho_q/2 - \delta)$ and note that by (8.234) one has $\lambda_{1/2,L} \leq L\varepsilon_1^{1/4}$ so that $U_{L,0} \subset \mathbf{T}_{qh/6} \times \mathbf{A}(2\lambda_{1/2,L}, \rho_q/2)$ and

$$\|\Sigma\|_{U_{L,0}} \lesssim qL^{-2}.$$

By Cauchy's estimates

$$\|\Sigma\|_{C^2(U_{L,\delta})} \lesssim q(\delta L)^{-2}. \tag{8.243}$$

Let us choose, $\delta = \varepsilon_1^{1/16}$ and

$$L = \varepsilon_1^{-7/32}. \tag{8.244}$$

We then have $L\varepsilon_1^{1/4} = \varepsilon_1^{(-7/32)+(8/32)} = \varepsilon_1^{1/32}$, $L^{-2}\delta^{-2} = \varepsilon_1^{(7/16)-(2/16)} = \varepsilon_1^{5/16}$, $L^{-2}\delta^{-3} = \varepsilon_1^{(7/16)-(3/16)} = \varepsilon_1^{1/4}$ hence

$$\|\Sigma\|_{C^2(\mathbf{T}_{qh/6}\times\mathbf{A}(2\varepsilon_1^{1/32},\rho_q/2))} \lesssim q\varepsilon_1^{1/4}. \tag{8.245}$$

Using Lemma 2.2 and Lemma 4.4, we see that $(\Sigma, \mathbf{T}_{qh/6} \times \mathbf{A}(2\varepsilon_1^{1/32}, \rho_q/2))$ has a C^2, σ-symmetric Whitney extension Σ^{Wh} such that

$$\overline{W} = f_{\Sigma^{Wh}}^{-1}, \qquad \overline{W}^{-1} = f_{\Sigma^{Wh}} \in \widetilde{\mathrm{Symp}}(\mathbf{T}_{qh/7} \times \mathbf{A}(3\varepsilon_1^{1/32}, \rho_q/3)) \tag{8.246}$$

and ($\|\overline{W}-id\|_{C^1}\lesssim \varepsilon_1^{-4/32}\varepsilon_1^{1/4}$)

$$\|\overline{W}-id\|_{C^1}\lesssim q\varepsilon_1^{1/8}. \tag{8.247}$$

On the other hand taking the derivative of (8.242) we have

$$\partial_\theta S(\theta,z)=g(\theta,H(z)) \tag{8.248}$$

and so S is a solution of the Hamilton-Jacobi equation

$$\overline{\Pi}(\theta,\frac{\partial S}{\partial\theta}(\theta,z))=\overline{\Pi}(\theta,g(\theta,H(z))) \tag{8.249}$$

$$=H^2(z) \qquad \text{(by (8.235))}. \tag{8.250}$$

Hence, the exact symplectic change of variable $\overline{W}=f_\Sigma^{-1}$

$$\overline{W}^{\,-1}=f_\Sigma:(\theta,w)\mapsto(\varphi,z)\Longleftrightarrow\begin{cases}w=\frac{\partial S}{\partial\theta}=w+\partial_\theta\Sigma(\theta,z)\\ \varphi=\frac{\partial S}{\partial z}=\theta+\partial_z\Sigma(\theta,z)\end{cases} \tag{8.251}$$

conjugates $\Phi_{\overline{\Pi}(\theta,w)}$ to $\Phi_{H(z)^2}$ since from (4.82)

$$\overline{\Pi}\circ\overline{W}=H^2 \qquad\Longleftrightarrow\qquad \overline{W}^{\,-1}\circ\Phi_{\overline{\Pi}}\circ\overline{W}=\Phi_{H^2}.$$

This concludes the proof. □

8.6. *Consequences on* $\Phi_{\widetilde{\Omega}}\circ f_{\widetilde{F}}$. — Let $G:(\theta,r)\mapsto(\theta,r+e_0(\theta))$ be the diffeomorphism introduced in Lemma 8.3 and

$$\widetilde{W}=G\circ\overline{W}. \tag{8.252}$$

We notice that $\widetilde{W}\in\widetilde{\mathrm{Symp}}_\sigma(\mathbf{T}_{qh/7}\times\mathbf{A}(3\varepsilon_1^{1/32},\rho_q/3))$ and that its image contains $G(\mathbf{T}_{qh/7}\times\mathbf{A}(3\varepsilon_1^{1/32},\rho_q/3))$ (see (8.246)); from (8.247) and (8.228) we have

$$\|\widetilde{W}-id\|_{C^1}\lesssim q\varepsilon_1^{1/8}. \tag{8.253}$$

Corollary **8.8.** — *One has*

$$\widetilde{W}^{-1}\circ\Phi_{\widetilde{\Omega}}\circ f_{\widetilde{F}^{res}}\circ\widetilde{W}=\Phi_{H^2}\circ f_{\widehat{F}^{vf}} \tag{8.254}$$

with

$$\|\widehat{F}^{vf}\|_{\mathbf{T}_{qh/8}\times\mathbf{A}(4\varepsilon_1^{1/32},\rho_q/4))}\lesssim\exp(-1/(q\overline{\rho})^{1/4})\|F\|_{h,\mathbf{D}(0,\overline{\rho})}. \tag{8.255}$$

Proof. — Recall that from (8.219) and the definition of $\widetilde{\Pi}$ (8.222))

$$\begin{aligned}\Phi_{\widetilde{\Omega}} \circ f_{\widetilde{F}^{res}} &= \Phi_{\widetilde{\Omega}+\widetilde{F}^{per}} \circ f_{\widetilde{F}^{vf}} \\ &= \Phi_{\widetilde{\Pi}} \circ f_{\widetilde{F}^{vf}}.\end{aligned}$$

By Lemma 8.3 and Proposition 8.7

$$G^{-1} \circ \Phi_{\widetilde{\Pi}} \circ G = \Phi_{\overline{\Pi}}, \qquad \overline{W}^{-1} \circ \Phi_{\overline{\Pi}} \circ \overline{W} = \Phi_{H^2}$$

hence

$$\widetilde{W}^{-1} \circ \Phi_{\widetilde{\Pi}} \circ \widetilde{W} = \Phi_{H^2}$$

and so

$$\widetilde{W}^{-1} \circ \Phi_{\widetilde{\Pi}} \circ f_{\widetilde{F}^{vf}} \circ \widetilde{W} = \Phi_{H^2} \circ f_{\widehat{F}^{vf}}, \qquad f_{\widehat{F}^{vf}} = \widetilde{W}^{-1} \circ f_{\widetilde{F}^{vf}} \circ \widetilde{W}$$

which is (8.254).

The estimate on $\widehat{F}^{vf}$ comes from (8.221) and (8.253). □

8.7. *Proof of Proposition 8.1: existence of Hamilton-Jacobi Normal Form.* — Let $\widetilde{W}$ be the diffeomorphism constructed in Corollary 8.8 (*cf.* (8.252)). The map Λ_q (defined in (8.214)) sends $((\mathbf{R}+i]-h/8,h/8[)/(2\pi/q)\mathbf{Z}) \times \mathbf{A}(\overline{c}; 4q^{-1}\varepsilon_1^{1/32}, \overline{\rho}/4)$ to $\mathbf{T}_{qh/8} \times \mathbf{A}(4\varepsilon_1^{1/32}, q\overline{\rho}/4)$. From (8.217), (8.254) one has

$$\widetilde{W}^{-1} \circ \Lambda_q \circ \Phi_{\overline{\Omega}} \circ f_{F^{res}_{j_q}} \circ \Lambda_q^{-1} \circ \widetilde{W} = \Phi_{H^2} \circ f_{\widehat{F}^{vf}}$$

hence

$$\begin{aligned}&(\Lambda_q^{-1} \circ \widetilde{W}^{-1} \circ \Lambda_q) \circ \Phi_{\overline{\Omega}} \circ f_{F^{res}_{j_q}} \circ (\Lambda_q^{-1} \circ \widetilde{W} \circ \Lambda_q) \\ &\quad = (\Lambda_q^{-1} \circ \Phi_{H^2} \circ \Lambda_q) \circ (\Lambda_q^{-1} \circ f_{\widehat{F}^{vf}} \circ \Lambda_q).\end{aligned}$$

Let W, $\Phi_{\mathring{\Omega}^{HJ}}$ and $f_{\mathring{F}^{vf}}$ be lifts by j_q (defined in (8.213)) of $\Lambda_q^{-1} \circ \widetilde{W} \circ \Lambda_q$, $\Lambda_q^{-1} \circ \Phi_{H^2} \circ \Lambda_q$ and $\Lambda_q^{-1} \circ f_{\widehat{F}^{vf}} \circ \Lambda_q$. Since $\Phi_{\overline{\Omega}} \circ f_{\overline{F}^{res}}$ is a lift by j_q of $\Phi_{\overline{\Omega}} \circ f_{F^{res}_{j_q}}$ one has for some $m \in \mathbf{Z}$ $(0 \le m \le q-1)$

$$W^{-1} \circ \Phi_{\overline{\Omega}} \circ f_{\overline{F}^{res}} \circ W = \Phi_{2\pi(m/q)r} \circ \Phi_{\mathring{\Omega}^{HJ}} \circ f_{\mathring{F}^{vf}}$$

where

(**8.256**) $$\mathring{\Omega}^{HJ}(r) = q^{-2}H^2(q(r-\overline{c})), \qquad \mathring{F}^{vf} = O(\widehat{F}^{vf}).$$

If we define

(**8.257**) $$f_{\mathring{F}^{cor}} = W^{-1} \circ f_{F^{cor}} \circ W, \qquad \widehat{F}^{cor} = O(F^{cor})$$

(**8.258**) $$g^{HJ} = g_{RNF} \circ W \qquad (g_{RNF} \text{ from } (8.211))$$

one has from (8.211) (note that W commutes with $\Phi_{2\pi(p/q)r}$)

$$(g^{\mathrm{HJ}})^{-1}\circ\Phi_{\Omega}\circ f_{\mathrm{F}}\circ g^{\mathrm{HJ}}=\Phi_{2\pi(p/q)r}\circ \mathrm{W}^{-1}\circ\Phi_{\overline{\Omega}}\circ f_{\overline{\mathrm{F}}^{per}}\circ \mathrm{W}\circ f_{\widehat{\mathrm{F}}^{cor}} \tag{8.259}$$
$$=\Phi_{2\pi(p/q)r}\circ\Phi_{2\pi(m/q)r}\circ\Phi_{\mathring{\Omega}^{\mathrm{HJ}}}\circ f_{\mathring{\mathrm{F}}^{vf}}\circ f_{\widehat{\mathrm{F}}^{cor}}$$
$$=:\Phi_{\Omega^{\mathrm{HJ}}}\circ f_{\mathrm{F}^{\mathrm{HJ}}}$$

with (see (8.256), (8.255), (8.212))

$$\Omega^{\mathrm{HJ}}\in\mathcal{O}_{\sigma}(\mathbf{A}(\overline{c};5q^{-1}\varepsilon_1^{1/32},\overline{\rho}/5)), \tag{8.260}$$

$$\Omega^{\mathrm{HJ}}(r)=2\pi((p+m)/q)r+q^{-2}\mathrm{H}^2(q(r-\overline{c})) \tag{8.261}$$

$$\mathrm{F}^{\mathrm{HJ}}=\mathring{\mathrm{F}}^{vf}+\widehat{\mathrm{F}}^{cor}+\mathfrak{O}_2(\mathring{\mathrm{F}}^{vf},\widehat{\mathrm{F}}^{cor})\in\mathcal{O}_{\sigma}(\mathbf{T}_{h/9}\times\mathbf{A}(\overline{c};5q^{-1}\varepsilon_1^{1/32},\overline{\rho}/5)) \tag{8.262}$$

$$\|\mathrm{F}^{\mathrm{HJ}}\|\lesssim\exp(-1/(q\overline{\rho})^{1/4})\overline{\varepsilon}. \tag{8.263}$$

With a slight abuse of notation, we can write $\mathrm{W}=\Lambda_q^{-1}\circ\widetilde{\mathrm{W}}\circ\Lambda_q$ and using (8.258), (8.252) and the definition of $\overline{\mathrm{W}}$ (*cf.* Proposition 8.7) we can write

$$g^{\mathrm{HJ}}=g_{\mathrm{RNF}}\circ\Lambda_q^{-1}\circ(\mathrm{G}\circ\overline{\mathrm{W}})\circ\Lambda_q\in\widetilde{\mathrm{Symp}}_{\sigma}(\mathbf{T}_{h/9}\times\mathbf{A}(\overline{c};5q^{-1}\varepsilon_1^{1/32},\overline{\rho}/5)).$$

The last inequality of (8.212) and (8.253) show that (remember (8.230))

$$\|g^{\mathrm{HJ}}-id\|_{\mathrm{C}^1}\lesssim q\varepsilon_1^{1/8}+q\overline{\varepsilon}^{1-}\lesssim q\overline{\varepsilon}^{1/8}\qquad(\varepsilon_1\lesssim q\overline{\rho}^{-1}\overline{\varepsilon}). \tag{8.264}$$

Note that since $\|g^{\mathrm{HJ}}-id\|_{\mathrm{C}^1}$, $\|f_{\mathrm{F}}-id\|$ (*cf.* (8.203)) and $\|f_{\mathrm{F}^{\mathrm{HJ}}}-id\|$ (*cf.* (8.263)) are $\ll 1/q$, the conjugation relation (8.259) shows that the integer m appearing in (8.261) must be equal to 0. Hence,

$$\Omega^{\mathrm{HJ}}(r)=2\pi(p/q)r+q^{-2}\mathrm{H}^2(q(r-\overline{c})). \tag{8.265}$$

Let us now check that one can choose Ω^{HJ} in $\widetilde{\mathcal{O}}_{\sigma}(\widehat{\mathrm{D}}\smallsetminus\check{\mathrm{D}})$ which satisfies a (2A, 2B)-twist condition. Indeed, from (8.239) and Cauchy's inequality (recall our choice (8.244) $\mathrm{L}=\varepsilon_1^{-7/32}$) we see that

$$\|q^{-2}\mathrm{H}^2(q(r-\overline{c}))-\gamma^{-2}(r-\overline{c})^2\|_{\mathrm{C}^3(\mathbf{T}_{h/9}\times\mathbf{A}(\overline{c};6q^{-1}\varepsilon_1^{1/32},\overline{\rho}/6))}$$
$$\lesssim q\varepsilon_1^{-3/32}\varepsilon_1^{7/16}\leq q\varepsilon_1^{11/32}.$$

We now apply Lemma 2.2: since $\varepsilon_1^{-6/32}\times\varepsilon_1^{11/32}\lesssim\varepsilon_1^{5/32}$, there exists a C^3 σ-symmetric Whitney extension with C^3-norm less that $q\varepsilon_1^{1/7}<\overline{\varepsilon}^{1/8}$ for $(q^{-2}\mathrm{H}^2(q(r-\overline{c}))-\gamma^{-2}(r-\overline{c})^2,\mathbf{T}_{h/9}\times\mathbf{A}(\overline{c};6q^{-1}\varepsilon_1^{1/32},\overline{\rho}/6))$. Using (8.265) and the inequality $\overline{\varepsilon}^{1/8}\ll(1/2)\min(\mathrm{A},\mathrm{B})$ (*cf.* (8.203)) we see that Ω^{HJ} has a Whitney extension (that we still denote Ω^{HJ}) such that

$$\Omega^{\mathrm{HJ}}\ \text{satisfies a (2A, 2B)}-\text{twist condition.} \tag{8.266}$$

We can now conclude the proof of Proposition 8.1. We define the disk $\check{D}$ of Proposition 8.1 as (*cf.* (8.230), (8.228))

$$\textbf{(8.267)} \qquad \begin{cases} \check{D} = \mathbf{D}(\check{c}, \check{\rho}) \subset \mathbf{D}(\bar{c}, \bar{\varepsilon}^{\,1/33}) \\ \check{c} = \bar{c}, \qquad \check{\rho} = 6\varepsilon_1^{1/32} = 6\|e_1\|_{C^0(\mathbf{T})}^{1/32} \leq \bar{\varepsilon}^{\,1/33} \qquad (\varepsilon_1 \lesssim q\bar{\rho}^{\,-1}\bar{\varepsilon}) \end{cases}$$

and the disk $\widehat{D}$ can be taken to be (recall $|\bar{c} - c| \lesssim \bar{\varepsilon}$)

$$\textbf{(8.268)} \qquad \widehat{D} = \mathbf{D}(c, \bar{\rho}/6) = \mathbf{D}(c, \widehat{\rho}).$$

With these notations, the set of conclusions (8.207)–(8.210) are consequences of (8.266), (8.259), (8.264) and (8.263). □

8.8. *Extending the linearizing map inside the hole.* — In general the previously defined maps g, Γ, H *are not* holomorphically defined on a *whole* disk but rather on an annulus with inner disk of radius $6\varepsilon_1^{1/32}$ where $\varepsilon_1 = \|e_1\|_{C^0(\mathbf{T})}$. In this subsection we quantify to which extent the domains of holomorphy of these maps can be extended if one knows that the frequency map Ω^{HJ} coincides on this annulus with a holomorphic function defined on a disk (containing the annulus).

Notation **8.9.** — *In the following we denote by* $C(0, t)$ *the circle of center 0 and radius* $t > 0$.

Proposition **8.10.** — *If there exists a holomorphic function* $\widetilde{\Xi}$ *defined on* $\mathbf{D}(0, \rho_q)$ *such that*

$$\textbf{(8.269)} \qquad \|\widetilde{\Xi} - H^2\|_{C(0,\rho_q/2)} \leq \nu$$

then

$$\varepsilon_1 = \|e_1\|_{C^0(\mathbf{T})} \lesssim \nu^{(1/6)-}.$$

We prove this proposition in Section 8.8.2.

We now take

$$s = 0. \qquad (cf.\ (8.231))$$

By (8.236) for $z \in \mathbf{A}(\lambda/2, \lambda/4)$, $|g(\theta, z)|$ compares to λ and thus from (8.235) and (8.227)

$$z^2 = \varpi(\theta)\Big(g(\theta, z)^2 - e_1(\theta)\Big) + O(g^3)$$

so that

$$g(\theta,z)=\Big(z^2/\varpi(\theta)+e_1(\cdot)+\mathrm{O}(g^3)\Big)^{1/2} \tag{8.270}$$

$$=\Big((z^2/\varpi(\theta))+e_1(\theta)\Big)^{1/2}+\mathrm{O}(\lambda^2). \tag{8.271}$$

Let's introduce

$$\widetilde{g}(\theta,z)=\left(\frac{z^2}{\varpi(\theta)}+e_1(\theta)\right)^{1/2} \tag{8.272}$$

$$\widetilde{\Gamma}(\cdot)=(2\pi)^{-1}\int_0^{2\pi}\widetilde{g}(\theta,\cdot)d\theta, \qquad \widetilde{\mathrm{H}}=\widetilde{\Gamma}^{-1} \tag{8.273}$$

where the inverse is with respect to composition. The functions $\widetilde{\Gamma}$ and $\widetilde{\mathrm{H}}$ are defined on $\{z\in\mathbf{C}, \mathrm{L}\varepsilon_1^{1/2}<|z|\}$ for some fixed $\mathrm{L}\gg 1$, independent of ε_1, satisfying

$$\mathrm{L}\le(\rho_q/2)\varepsilon_1^{-1/2} \tag{8.274}$$

(we take here $s=0$, *cf.* (8.231)).

8.8.1. *Computation of a residue.*

Lemma **8.11.** — *For any circle* $\mathrm{C}(0,t)$ *centered at 0 with* $\mathrm{L}\varepsilon_1^{1/2}<t<\rho_q/2$ *one has*

$$\frac{1}{2\pi i}\int_{\mathrm{C}(0,t)} z\widetilde{\mathrm{H}}(z)^2dz=(\gamma/4)\int_{\mathbf{T}}\varpi(\theta)^{3/2}e_1(\theta)^2\frac{d\theta}{2\pi}$$

where $\gamma=(2\pi)^{-1}\int_0^{2\pi}\varpi(\theta)^{-1/2}d\theta$.

Proof. — We compute the expansion of $\widetilde{g}(\theta,\cdot)$ (*cf.* (8.272)) into Laurent series: on $\mathbf{C}\setminus\mathbf{D}(0,\mathrm{L}\varepsilon_1^{1/2})$:

$$\begin{aligned}\widetilde{g}(\theta,z)&=(z/\varpi(\theta)^{1/2})\Big(1+\varpi(\theta)e_1(\theta)z^{-2}\Big)^{1/2}\\&=(z/\varpi(\theta)^{1/2})\Big(1+\frac{1}{2}\varpi(\theta)e_1(\theta)z^{-2}-\frac{1}{8}(\varpi(\theta)e_1(\theta))^2z^{-4}+\mathrm{O}(z^{-6})\Big)\\&=\frac{z}{\varpi(\theta)^{1/2}}+\frac{1}{2}\varpi(\theta)^{1/2}e_1(\theta)z^{-1}-\frac{1}{8}\varpi(\theta)^{3/2}e_1(\theta)^2z^{-3}+\mathrm{O}(z^{-5}).\end{aligned}$$

As a consequence since $\widetilde{\Gamma}(z) = (2\pi)^{-1}\int_0^{2\pi}\widetilde{g}(\theta,z)d\theta$ we have with the notation $\gamma = (2\pi)^{-1}\int_0^{2\pi}\varpi(\theta)^{-1/2}d\theta$ the identity

$$\widetilde{\Gamma}(z) = \gamma(z + a_{-1}z^{-1} + a_{-3}z^{-3}) + O(z^{-5})$$

where

$$a_{-1} = \gamma^{-1}(1/2)(2\pi)^{-1}\int_0^{2\pi}\varpi(\theta)^{1/2}e_1(\theta)d\theta \tag{8.275}$$
$$a_{-3} = \gamma^{-1}(-1/8)(2\pi)^{-1}\int_0^{2\pi}\varpi(\theta)^{3/2}e_1(\theta)^2 d\theta.$$

By our choice (8.229) we have $a_{-1} = 0$ and we can thus write

$$\widetilde{\Gamma} = \Lambda_\gamma \circ (id + u) \tag{8.276}$$

where $\Lambda_\gamma z = \gamma z$ and

$$u(z) = a_{-3}z^{-3} + O(z^{-5}).$$

If v is defined by

$$(id + u) \circ (id + v) = id$$

we have

$$v(z) = -a_{-3}z^{-3} + O(z^{-4})$$

and therefore

$$\begin{aligned}(z + v(z))^2 &= (z - a_{-3}z^{-3} + O(z^{-4}))^2 \\ &= z^2 - 2a_{-3}z^{-2} + O(z^{-3}).\end{aligned} \tag{8.277}$$

Now since $\widetilde{H}$ is the inverse for the composition of $\widetilde{\Gamma}$ (*cf.* (8.273)), $z = (\widetilde{\Gamma} \circ \widetilde{H})(z)$, we have by (8.276) $\widetilde{H} = (id + u)^{-1} \circ \Lambda_\gamma^{-1} = (id + v) \circ \Lambda_\gamma^{-1}$ and we get by (8.277)

$$\widetilde{H}(z)^2 = \gamma^{-2}z^2 - 2a_{-3}\gamma^2 z^{-2} + O(z^{-3})$$

and thus

$$z\widetilde{H}(z)^2 = \gamma^{-2}z^3 - 2a_{-3}\gamma^2 z^{-1} + O(z^{-2}).$$

Hence by Cauchy's formula and (8.275), for any circle $C(0,t)$, $L\varepsilon_1^{1/2} < t < \rho_q/2$:

$$\frac{1}{2\pi i}\int_{C(0,t)} z\widetilde{H}(z)^2 dz = -2a_{-3}\gamma^2$$
$$= (\gamma/4)\int_{\mathbf{T}} \varpi(\theta)^{3/2} e_1(\theta)^2 \frac{d\theta}{2\pi}. \qquad \square$$

8.8.2. *Proof of Proposition 8.10.*

Lemma **8.12.** — *Let* $L\varepsilon_1^{1/2} \leq \lambda < \rho_q/2$, $L \gg 1$ *(independent of* ε_1*). One has for* $z \in \mathbf{A}(\lambda/4, \lambda/2)$

$$|H(z)^2 - \widetilde{H}(z)^2| \lesssim \lambda^3. \tag{8.278}$$

Proof. — For $z \in \mathbf{A}(\lambda/4, \lambda/2)$, $\theta \in \mathbf{T}$ one has by (8.271), (8.272)

$$|g(\theta, z) - \widetilde{g}(\theta, z)| \lesssim \lambda^2$$

so (*cf.* (8.237), (8.273))

$$|\Gamma(z) - \widetilde{\Gamma}(z)| \lesssim \lambda^2. \tag{8.279}$$

On the other hand, from Lemma L.1

$$e^{-3/L^2} \leq \left|\frac{\widetilde{g}(\theta, z) - \widetilde{g}(\theta, z')}{z - z'}\right| \leq e^{2/L^2}$$

hence

$$e^{-3/L^2} \leq \left|\frac{\widetilde{\Gamma}(z) - \widetilde{\Gamma}(z')}{z - z'}\right| \leq e^{2/L^2}. \tag{8.280}$$

Since $z = \Gamma(H(z)) = \widetilde{\Gamma}(\widetilde{H}(z))$ and $H(z), \widetilde{H}(z) \asymp z$ (*cf.* (8.239)), one has from (8.279)

$$|\widetilde{\Gamma}(H(z)) - \widetilde{\Gamma}(\widetilde{H}(z))| \lesssim \lambda^2$$

and so from (8.280)

$$|\widetilde{H}(z) - H(z)| \lesssim \lambda^2.$$

Since from (8.239) $|\widetilde{H}(z) + H(z)| \lesssim \lambda$ we thus have

$$|\widetilde{H}(z)^2 - H(z)^2| \lesssim \lambda^3. \qquad \square$$

We recall that $\varepsilon_1 = \|e_1\|_{C^0(\mathbf{T})}$. The function $\widetilde{\Xi} - H^2$ satisfies (*cf.* (8.269), (8.239))

$$\|\widetilde{\Xi} - H^2\|_{C(0,\rho_q/2)} \lesssim \nu, \qquad \|\widetilde{\Xi} - H^2\|_{C(0,L\varepsilon_1^{1/2})} \lesssim 1.$$

Let $M > 5$ and

$$\lambda_M := (\rho_q/2)^{1/M}(L\varepsilon_1^{1/2})^{1-1/M} \leq (L\varepsilon_1^{1/2})^{1-1/M} \tag{8.281}$$

(we can assume $\rho_q \leq 1$). By the Three Circles Theorem,

$$\|\widetilde{\Xi} - H^2\|_{C(0,\lambda_M)} \lesssim \nu^{1/M}.$$

Lemma 8.12 tells us that

$$\|\widetilde{\Xi} - \widetilde{H}^2\|_{C(0,\lambda_M)} \lesssim \nu^{1/M} + \lambda_M^3$$

hence for any z in the circle $C(0, \lambda_M)$

$$|z\widetilde{\Xi}(z) - z\widetilde{H}^2(z)| \lesssim \lambda_M(\nu^{1/M} + \lambda_M^3)$$

and

$$\left|\frac{1}{2\pi i}\int_{C(0,\lambda_M)} (z\widetilde{\Xi}(z) - z\widetilde{H}^2(z))dz\right| \lesssim \lambda_M^2(\nu^{1/M} + \lambda_M^3).$$

Since $z \mapsto z\widetilde{\Xi}(z)$ is holomorphic on $D(0, 2\lambda_M)$, $\int_{C(0,\lambda_M)} z\widetilde{\Xi}(z)dz = 0$ and by Lemma 8.11 we get

$$\int_{\mathbf{T}} \varpi^{3/2}(\theta)e_1(\theta)^2 d\theta \lesssim \lambda_M^2(\nu^{1/M} + \lambda_M^3).$$

Since $\varpi(\theta) \gtrsim 1$ this gives

$$\int_{\mathbf{T}} e_1(\theta)^2 d\theta \lesssim \lambda_M^2(\nu^{1/M} + \lambda_M^3)$$

hence remembering (8.281)

$$\begin{aligned}\|e_1\|_{L^2(\mathbf{T})} &\lesssim \lambda_M \nu^{1/(2M)} + \lambda_M^{5/2}\\ &\lesssim L^{(1-1/M)}\varepsilon_1^{(1/2)(1-1/M)}\nu^{1/(2M)} + L^{(5/2)(1-1/M)}\varepsilon_1^{(5/4)(1-1/M)}.\end{aligned}$$

If we define

$$\delta_M = L^{(5/2)(1-1/M)}\varepsilon_1^{(5/4)(1-1/M)-1}, \qquad \mu_M = L^{(1-1/M)}\varepsilon_1^{(1/2)(1-1/M)}\nu^{1/(2M)}$$

this can be written (recall that $\|e_1\|_{C^0(\mathbf{T})} = \varepsilon_1 \lesssim q\overline{\rho}^{\,-1}\overline{\varepsilon}$) for some $C > 0$

$$\|e_1\|_{L^2(\mathbf{T})} \leq C\delta_M\|e_1\|_{C^0(\mathbf{T})} + C\mu_M$$

and we are in position to apply Lemma M.2 (our choice $M > 5$ implies that for some $\beta > 0$, $\delta_M \le \varepsilon_1^{2\beta} \ll 1$):

$$\begin{aligned}\varepsilon_1 = \|e_1\|_{C^0(\mathbf{T})} &\le (\mu_M/\delta_M) + Ch^{-1}\exp(-h/(C\delta_M^2))q\overline{\rho}^{\,-1}\overline{\varepsilon} \\ &\lesssim (\mu_M/\delta_M) + \exp(-(1/\varepsilon_1)^{\beta}) \qquad (\beta > 0) \\ &\lesssim (\mu_M/\delta_M) + (1/2)\varepsilon_1\end{aligned}$$

which gives

$$\varepsilon_1 \lesssim L^{-(3/2)(1-1/M)}\varepsilon_1^{1-((3/4)(1-1/M))}\nu^{1/(2M)}$$

or equivalently

$$\varepsilon_1^{(3/4)(1-1/M)} \lesssim L^{-(3/2)(1-1/M)}\nu^{1/(2M)}$$

and taking $M = 5+$, one finally gets:

$$\begin{aligned}\varepsilon_1 &\lesssim L^{-(2-)}\nu^{(1/6)-} \\ &\le \nu^{(1/6)-}.\end{aligned}$$

This completes the proof of Proposition 8.10. □

8.9. *Proof of Proposition 8.1: the Extension Property*. — From (8.265) we see that if there exists a holomorphic function Ξ defined on $\widehat{D}$ such that

$$\|\Xi - \Omega_D^{HJ}\|_{(4/5)\widehat{D}\setminus(1/5)\widehat{D}} \lesssim \nu$$

there exists a holomorphic function $\widetilde{\Xi}$ defined on $\mathbf{D}(0,\rho_q)$ (recall that $\rho_q = q\overline{\rho}/3$ *cf.* (8.226)) such that

$$\|\widetilde{\Xi} - H^2\|_{C(0,\rho_q/2)} \lesssim \nu$$

and thus by Proposition 8.10

$$\varepsilon_1 = \|e_1\|_{C^0(\mathbf{T})} \lesssim \nu^{(1/6)-}.$$

Now (8.267) shows that the conclusion of Proposition 8.1 holds with $\check{D} = D(c, \nu^{1/200})$. □

9. Comparison Principle for Normal Forms

In this section, if $0 \le \rho_1 < \rho_2$, we denote by $\mathbf{A}(c;\rho_1,\rho_2)$ the annulus $\{z \in \mathbf{C},\ \rho_1 \le |z - c| < \rho_2\}$ (it is thus the disk $\mathbf{D}(c,\rho_2)$ if $\rho_1 = 0$).

Proposition **9.1** *((AA) Case). — There exist positive constants* $\overline{C}, \overline{a}_4, \overline{a}_5$ *for which the following holds. Let* $0 < \rho_1 < \rho_2$ *(resp.* $\rho_1 = 0 < \rho_2$*),* $\varepsilon, \nu > 0$ *and for* $j = 1, 2$, $\Omega_j \in \widetilde{\mathcal{O}}_\sigma(\mathbf{A}(c; \rho_1, \rho_2))$, $F_j \in \mathcal{O}_\sigma(W_{h,\mathbf{A}(c;\rho_1,\rho_2)})$, $g_j \in \widetilde{\mathrm{Symp}}_\sigma(W_{h,\mathbf{A}(c;\rho_1,\rho_2)})$ *such that:* Ω_1, Ω_2 *satisfy an* (A, B)*-twist condition (2.59) and*

$$\|g_j - id\|_{C^1} \le \varepsilon < \overline{C}^{-1} h \tag{9.282}$$
$$\|F_j\|_{W_{h,\mathbf{A}(c;\rho_1,\rho_2)}} \le \nu$$

and on $g_1(\mathbf{A}(c; \rho_1, \rho_2)) \cap g_2(\mathbf{A}(c; \rho_1, \rho_2))$ *one has*

$$g_1 \circ \Phi_{\Omega_1} \circ f_{F_1} \circ g_1^{-1} = g_2 \circ \Phi_{\Omega_2} \circ f_{F_2} \circ g_2^{-1}.$$

Then, if $\delta > 0$ *satisfies*

$$\overline{C}\varepsilon \le \delta/4 < (\rho_2 - \rho_1) \quad \textit{and} \quad \overline{C}\delta^{-\overline{a}_4}\nu < 1, \tag{9.283}$$

there exists $\gamma \in \mathbf{R}$, $|\gamma| \le \overline{C}\varepsilon$ *such that one has*

$$\|\partial\Omega_1(\cdot + \gamma) - \partial\Omega_2\|_{\mathbf{A}(c;\rho_1+\delta,\rho_2-\delta)} \le \overline{C}\delta^{-\overline{a}_5}\nu. \tag{9.284}$$
$$(\textit{resp.} \quad \|\partial\Omega_1(\cdot + \gamma) - \partial\Omega_2\|_{\mathbf{D}(c,\rho_2-\delta)} \le \overline{C}\delta^{-\overline{a}_5}\nu.)$$

Furthermore, if g_1 *and* g_2 *are exact symplectic on* $M_{\mathbf{R}}$*, one can choose* $\gamma = 0$.

Proof. — We only treat the case $\rho_1 > 0$ (the case $\rho_1 = 0$ is done similarly).

From (9.282) we see that there exists $C > 0$ such that one has on $W_1 := \mathbf{T}_{h-C\varepsilon} \times \mathbf{A}(c; \rho_1 + C\varepsilon, \rho_2 - C\varepsilon)$

$$g \circ \Phi_{\Omega_1} = \Phi_{\Omega_2} \circ g \circ f_F \tag{9.285}$$

where

$$g := g_2^{-1} \circ g_1 \in \widetilde{\mathrm{Symp}}_\sigma(W_1)$$
$$F \in \mathcal{O}(W_1), \qquad f_F := g^{-1} \circ f_{F_2} \circ g \circ f_{F_1}^{-1}, \qquad \|F\|_{W_1} \lesssim \nu.$$

We write

$$g(\theta, r) = (\theta + u(\theta, r), r + v(\theta, r)) \tag{9.286}$$

and we introduce the notations $\omega_i = \partial\Omega_i$, $i = 1, 2$ (we drop the usual factor $(2\pi)^{-1}$). We have

$$g \circ \Phi_{\Omega_1}(\theta, r) = (\theta + \omega_1(r) + u(\theta + \omega_1(r), r), r + v(\theta + \omega_1(r), r))$$

and

$$\Phi_{\Omega_2} \circ g = (\theta + u(\theta, r) + \omega_2(r + v(\theta, r)), r + v(\theta, r))$$

We thus have on $\mathrm{W}_2 := \mathbf{T}_{h-\mathrm{C}\varepsilon-\mathrm{B}\rho_2-\delta} \times \mathbf{A}(c; \rho_1 + \mathrm{C}\varepsilon + \delta, \rho_2 - \mathrm{C}\varepsilon - \delta)$

$$(\mathbf{9.287}) \qquad \begin{cases} \omega_2(r + v(\theta, r)) - \omega_1(r) = \mathrm{I} + u(\theta + \omega_1(r), r) - u(\theta, r) \\ v(\theta + \omega_1(r), r) - v(\theta, r) = \mathrm{II} \end{cases}$$

with $\max(\|\mathrm{I}\|_{\mathrm{W}_2}, \|\mathrm{II}\|_{\mathrm{W}_2}) = \mathrm{O}(\delta^{-b}\nu)$. We observe that from the twist assumption on Ω_1 there exists a set $\mathrm{R} \subset \mathbf{A}(c; \rho_1 + \mathrm{C}\varepsilon + \delta, \rho_2 - \mathrm{C}\varepsilon - \delta)$ of Lebesgue measure $\lesssim \delta^2$, which is a countable union of disks centered on the real axis, such that one has for any $r \in \mathbf{A}(c; \rho_1 + \mathrm{C}\varepsilon + \delta, \rho_2 - \mathrm{C}\varepsilon - \delta) \setminus \mathrm{R}$ and any $k \in \mathbf{Z}^*$

$$(\mathbf{9.288}) \qquad \min_{l \in \mathbf{Z}} |\omega_1(r) - 2\pi \frac{l}{k}| \geq \frac{\delta^2}{k^3}$$

so that the second identity in (9.287) gives for any $r \in \mathbf{A}(c; \rho_1 + \mathrm{C}\varepsilon + \delta, \rho_2 - \mathrm{C}\varepsilon - \delta) \setminus \mathrm{R}$ the following inequality on $\mathbf{T}_{h_1-2\delta}$ (where $h_1 = h - \mathrm{C}\varepsilon - \mathrm{B}\rho_2$)

$$(\mathbf{9.289}) \qquad \|v(\cdot, r) - \int_{\mathbf{T}} v(\theta, r)d\theta\|_{h_1-2\delta} \lesssim \delta^{-3}\delta^{-b}\nu.$$

We now notice that there exists $0 \leq t \leq \delta^2$ such that $\mathrm{R} \cap \partial\mathbf{A}(c; \rho_1 + \mathrm{C}\varepsilon + \delta + t, \rho_2 - \mathrm{C}\varepsilon - \delta - t) = \emptyset$. The maximum principle applied, for any $\varphi \in \mathbf{T}_{h_1-2\delta}$, to the holomorphic function $v(\varphi, \cdot) - \int_{\mathbf{T}} v(\theta, \cdot)d\theta$ defined on $\mathbf{A}(c; \rho_1 + \mathrm{C}\varepsilon + \delta + t, \rho_2 - \mathrm{C}\varepsilon - \delta - t)$ shows that (9.289) holds for any $r \in \mathbf{A}_{2\delta} := \mathbf{A}(c; \rho_1 + \mathrm{C}\varepsilon + 2\delta, \rho_2 - \mathrm{C}\varepsilon - 2\delta)$. We thus have

$$(\mathbf{9.290}) \qquad \|\partial_\theta v\|_{h_1-3\delta, \mathbf{A}_{3\delta}} = \mathrm{O}(\delta^{-(4+b)}\nu).$$

Taking the ∂_θ derivative of the first line of (9.287) and using the previous inequality show that (from now on the value of b may change from line to line)

$$\partial_\theta u(\theta + \omega_1(r), r) - \partial_\theta u(\theta, r) = \mathrm{O}(\delta^{-b}\nu).$$

By the same argument used to establish (9.290) we get

$$(\mathbf{9.291}) \qquad \|\partial_\theta u\|_{h_1-4\delta, \mathbf{A}_{4\delta}} = \mathrm{O}(\delta^{-b}\nu)$$

(we have used the fact that $\int_{\mathbf{T}} \partial_\theta u(\theta, r)d\theta = 0$). Since g is symplectic on W_1, $\det \mathrm{D}g(\theta, r) \equiv 1$ hence

$$(1 + \partial_\theta u(\theta, r))(1 + \partial_r v(\theta, r)) - \partial_r u(\theta, r)\partial_\theta v(\theta, r) = 1$$

and in view of (9.290), (9.291)

$$\|\partial_r v\|_{h_1-4\delta, \mathbf{A}_{4\delta}} = \mathrm{O}(\delta^{-b}\nu)$$

which combined with (9.290) implies,

$$\|v-\gamma\|_{h_1-4\delta,\mathbf{A}_{4\delta}}=\mathrm{O}(\delta^{-b}\nu),\qquad \gamma=v(0,0)\in\mathbf{R}. \tag{9.292}$$

The first equation of (9.287) implies that

$$\|\omega_2(\cdot+\gamma)-\omega_1(\cdot)\|_{\mathbf{A}_{4\delta}}=\mathrm{O}(\delta^{-b}\nu)$$

which is the first conclusion of the Proposition (9.284).

If g_1 and g_2 are exact symplectic, g is also exact symplectic and one can write $g=f_Z$ for some $Z=\mathfrak{O}_1(g-id)$ which means $g(\theta,r)=(\varphi,R)$ if and only if $r=R+\partial_\theta Z(\theta,R)$, $\varphi=\theta+\partial_R Z(\theta,R)$. In particular

$$r=r+v(\theta,r)+\partial_\theta Z(\theta,r+v(\theta,r))$$

and since

$$\frac{d}{d\theta}Z(\theta,r+v(\theta,r))=\partial_\theta Z(\theta,r+v(\theta,r))+\partial_R Z(\theta,r+v(\theta,r)\partial_\theta v(\theta,r)$$

we get from (9.290)

$$v(\theta,r)=-\frac{d}{d\theta}Z(\theta,r+v(\theta,r))+\mathrm{O}(\delta^{-b}\nu)$$

which after integration in θ yields

$$\int_{\mathbf{T}}v(\theta,r)d\theta=\mathrm{O}(\delta^{-b}\nu).$$

We can now conclude from (9.292) that

$$\gamma=\mathrm{O}(\delta^{-b}\nu).$$

In particular, taking $\gamma=0$ does not affect the estimate (9.284). □

Proposition **9.2** *((CC)-Case). — Under the assumptions of the previous Proposition 9.1:*

(1) If $c=0$, $\rho=\rho_2$, $\rho_1=0$ and g_1,g_2 are exact symplectic then

$$\|\partial\Omega_1(\cdot)-\partial\Omega_2(\cdot)\|_{\mathbf{D}(0,\rho_2-\delta)}\le C\delta^{-\bar{a}_5}\nu.$$

(2) If $\rho_2<|c|/4$ then all the conclusions of the previous Proposition 9.1 are valid.

Proof. — The proof of Item (2) follows from Item (2) of Lemma K.1 of the Appendix applied to Proposition 9.1.

So we concentrate on the proof of Item (1), $c=0$, $\rho=\rho_2$, $\rho_1=0$. We use the symplectic change of coordinates of Section K $(\theta,r)=\psi_\pm^{-1}(z,w)$,

$$\psi_\pm:\mathbf{T}_h\times\Delta_\alpha^\pm(0,\rho)\to W_{h,\Delta_\alpha^\pm(0,\rho)}\cap\{e^{-2h}<|z|/|w|<e^{2h}\}$$

where $\alpha<\pi/10$. Setting $g_{j,\pm}=\psi_\pm^{-1}\circ g_j\circ\psi_\pm$, $g_\pm=g_{2,\pm}^{-1}\circ g_{1,\pm}$, $g_\pm(\theta,r)=(\theta+u_\pm(\theta,r),r+v_\pm(\theta,r))$ we are then reduced to the preceding situation where g is replaced by $g_\pm$, the annulus $\mathbf{A}(c_2;\rho_1,\rho_2)$ is replaced by the angular sector $\Delta^\pm_{\alpha+4\delta}(\rho-4\delta)$ and h by $h-4\delta$, so that (9.287) holds on $\mathbf{T}_{h-4\delta}\times\Delta^\pm_{\alpha-4\delta}(\rho-4\delta)$. Like in the previous case, one can find $0\le t\le\delta^2$ such that the Diophantine condition (9.288) holds for any $r\in\Delta^\pm_{\alpha+4\delta}(0;t,\rho-4\delta-t):=\Delta^\pm_{\alpha+4\delta}(\rho-4\delta)\cap\mathbf{A}(0;t,\rho-4\delta-t)$. Still by the Maximum Principle (9.290) holds on $\mathbf{T}_{h-5\delta}\times\Delta^\pm_{\alpha+5\delta}(0;t,\rho-5\delta-t)$ with v replaced by $v_\pm$ and one can conclude as we've done before that (9.291) holds with u replaced by $u_\pm$ as well. Finally this gives the existence of $\gamma_\pm=v_\pm(0,0)\in\mathbf{R}$ such that on $\Delta^\pm_{\alpha+5\delta}(0;t,\rho-5\delta-t)$

$$\|\omega_2(\cdot+\gamma_\pm)-\omega_1(\cdot)\|_{\mathbf{A}_{4\delta}}=O(\delta^{-b}\nu). \tag{9.293}$$

Now, if g_1 and g_2 are exact symplectic the same is true for $g_{1,\pm},g_{2,\pm}$ (*cf.* Remark 4.2) and hence $g_\pm$ is also exact symplectic; we can thus prove, like in the proof of Proposition 9.1, that $\pm\gamma=O(\delta^{-b}\nu)$. We can hence assume that $\gamma_\pm=0$ in equation (9.293). Since $\alpha<\pi/10$ we deduce that on $\mathbf{A}(0;t,\rho-5\delta-t)=\Delta^+_{\alpha+5\delta}(0;t,\rho-5\delta-t)\cup\Delta^-_{\alpha+5\delta}(0;t,\rho-5\delta-t)$ one has

$$\|\omega_2(\cdot)-\omega_1(\cdot)\|_{\mathbf{A}(0;t,\rho-5\delta-t)}=O(\delta^{-b}\nu).$$

But $\omega_1,\omega_2\in\mathcal{O}(\mathbf{D}(0,\rho))$, hence by the Maximum Principle

$$\|\omega_2(\cdot)-\omega_1(\cdot)\|_{\mathbf{D}(0,\rho-5\delta-t)}=O(\delta^{-b}\nu).$$ □

10. Adapted Normal Forms: ω_0 Diophantine

Recall that for $\tau\ge1$, $\kappa>0$

$$DC(\kappa,\tau)=\{\omega_0\in\mathbf{R},\ \forall\,k\in\mathbf{Z}^*,\ \min_{l\in\mathbf{Z}}|\omega_0-\frac{l}{k}|\ge\frac{\kappa}{|k|^{1+\tau}}\}$$

$$DC(\tau)=\bigcup_{\kappa>0}DC(\kappa,\tau).$$

Let $h>0$, $0<\overline{\rho}<1$, $A,B\ge1$, $\Omega\in\widetilde{\mathcal{O}}_\sigma(e^{10h}\mathbf{D}(0,\overline{\rho}))$, $F\in\mathcal{O}_\sigma(e^{10h}W_{h,\mathbf{D}(0,\overline{\rho})})$ such that

$$\forall\,r\in\mathbf{R},\ A^{-1}\le(2\pi)^{-1}\partial^2\Omega(r)\le A,\quad\text{and}\ \|(2\pi)^{-1}D^3\Omega\|_{\mathbf{C}}\le B \tag{10.294}$$

$$\omega_0:=(2\pi)^{-1}\partial\Omega(0)\in DC(\kappa,\tau)\subset DC(\tau) \tag{10.295}$$

$$\forall\, 0<\rho\le\overline{\rho},\qquad \|F\|_{e^{10h}W_{h,\mathbf{D}(0,\rho)}}\le\rho^{m},\tag{10.296}$$

$$\text{where}\quad m=\max(\overline{a}_{1,\tau},\overline{a}_2+4,\overline{a}_3,\overline{a}_4,2b_\tau+10)\tag{10.297}$$

($\overline{a}_{1,\tau},\overline{a}_2,\overline{a}_3,\overline{a}_4$ are the constants appearing in Propositions 6.5, 7.1, 8.1, 9.1 and b_τ is defined by (6.152)).

We as usual denote $\omega=(1/2\pi)\partial\Omega$ $(\omega(0)=\omega_0)$.

10.1. *Adapted KAM domains.* — We use in this section the notations of Section 7, in particular we denote

$$\overline{\varepsilon}:=\max_{0\le j\le 3}\|D^jF\|_{h,\mathbf{D}_{\overline{\rho}}}\le\overline{\rho}^{\,\overline{a}_2}.\tag{10.298}$$

Assumption (10.296) allows us to apply Proposition 7.1 on the existence of a KAM Normal Form on the domain $W_{2h,\mathbf{D}(0,\overline{\rho})}$. We can thus define holed domains U_n and maps F_n, Ω_n, $g_{m,n}$ satisfying the conclusions of Proposition 7.1.

10.1.1. *Definition of the domains* $U_i^{(\rho)}$. — Let $0<\beta\ll_\tau 1$ and $\mu\in]1,1+1/\tau[$ such that

$$\mu\underset{\text{defin.}}{=}\left(1+\frac{1}{\tau}\right)(1-\beta)\in]1,1+1/\tau[.\tag{10.299}$$

We define for $\rho<\overline{\rho}/4$ two indices $i_-(\rho),\ i_+(\rho)\in\mathbf{N}$ as follows:

$$i_-(\rho)\underset{\text{defin.}}{=}\max\{i\ge 1,\ \mathbf{D}(0,2\rho)\cap U_i=\mathbf{D}(0,2\rho)\}\tag{10.300}$$

and

$$\begin{cases} i_+(\rho)\text{ is the unique index such that}\\ (N_{i_-(\rho)})^{\mu}\le N_{i_+(\rho)}<(4/3)^{\mu}(N_{i_-(\rho)})^{\mu}\le N^2_{i_-(\rho)}.\end{cases}\tag{10.301}$$

We also define $\iota(\rho)\in\mathbf{R}_+^*$ by

$$\begin{cases}\iota(\rho)\in\mathbf{R}_+^*\text{ such that}\\ \rho=(N_{i_-(\rho)})^{-\iota(\rho)},\qquad N_{i_-(\rho)}=\rho^{-1/\iota(\rho)}.\end{cases}\tag{10.302}$$

The next lemma shows how $N_{i_-(\rho)}$ and $N_{i_+(\rho)}$ compare with ρ.

Lemma **10.1.** — *One has*

$$(1+1/\tau)+O(|\ln\rho|^{-1})\le\iota(\rho)\le(1+\tau)+O(|\ln\rho|^{-1}).\tag{10.303}$$

In particular,

$$N_{i_+(\rho)}\asymp\rho^{-\mu/\iota(\rho)},\tag{10.304}$$

where for $\rho \ll_\beta 1$

$$\frac{1}{\tau} - 2\beta \le \frac{\mu}{\iota(\rho)} \le 1 - (\beta/2). \tag{10.305}$$

Proof. — To prove (10.303) we just have to check that

$$(\mathrm{N}_{i_-(\rho)})^{-(1+\tau)} \lesssim \rho \lesssim (\mathrm{N}_{i_-(\rho)})^{-(1+1/\tau)}. \tag{10.306}$$

See the details in Appendix I.1. □

We shall say that the domains U_i, $i_-(\rho) \le i \le i_+(\rho)$, are ρ-adapted KAM domains. For $t > 0$ and $i_-(\rho) \le i \le i_+(\rho)$ we define

$$\mathrm{U}_i^{(t)} = \mathrm{U}_i \cap \mathbf{D}(0,t), \quad \mathcal{D}_t(\mathrm{U}_i) = \mathcal{D}(\mathrm{U}_i^{(t)}) = \{\mathrm{D} \in \mathcal{D}(\mathrm{U}_i),\ \mathrm{D} \cap \mathbf{D}(0,t) \neq \emptyset\}$$

U_i being the domains of Proposition 7.1 and where as usual $\mathcal{D}(\mathrm{U})$ denotes the holes of the holed domain U (see Section 2.3.1). By (7.193)

$$\begin{cases} \mathrm{U}_i^{(t)} := \mathrm{U}_i \cap \mathbf{D}(0,t) = \mathbf{D}(0,t) \setminus \bigcup_{j=1}^{i-1} \bigcup_{(k,l)\in \mathrm{E}_j} \mathbf{D}(c_{l/k}^{(j)}, s_{j,i-1}\mathrm{K}_j^{-1}), \\ s_{j,i-1} = e^{\sum_{m=j}^{i-1} \delta_m} \in [1,2] \end{cases} \tag{10.307}$$

where

$$\mathrm{E}_j \subset \{(k,l) \in \mathbf{Z}^2,\ 0 < k < \mathrm{N}_j,\ 0 \le |l| \le \mathrm{N}_j\}, \qquad \omega_j(c_{l/k}^{(j)}) = l/k.$$

One can in fact in formula (10.307) restrict the union indexed by j to the set $j \in [i_-(\rho), i-1] \cap \mathbf{N}$; *cf.* Lemma I.1 of Appendix I.

One can also describe $\mathrm{U}_i^{(t)}$ by means of its holes:

$$\mathrm{U}_i^{(t)} := \mathrm{U}_i \cap \mathbf{D}(0,t) = \mathbf{D}(0,t) \setminus \bigcup_{\mathrm{D} \in \mathcal{D}_t(\mathrm{U}_i)} \mathrm{D} \tag{10.308}$$

this decomposition being minimal. In particular, if $\mathrm{D}, \mathrm{D}' \in \mathcal{D}_t(\mathrm{U}_i)$ the inclusions $\mathrm{D} \subset \mathrm{D}'$, $\mathrm{D}' \subset \mathrm{D}$ do not occur.

Proposition **10.2.** — *Let* $i_-(\rho) \le i' < i \le i_+(\rho)$.

(1) The holes $\mathrm{D} \in \mathcal{D}_{(3/2)\rho}(\mathrm{U}_i)$ *are pairwise disjoint.*

(2) If $\mathrm{D} \in \mathcal{D}_{(3/2)\rho}(\mathrm{U}_i)$, $\mathrm{D}' \in \mathcal{D}_{(3/2)\rho}(\mathrm{U}_{i'})$ *one has either* $\mathrm{D} \cap \mathrm{D}' = \emptyset$ *or* $\mathrm{D}' \subset \mathrm{D}$.

(3) The number of holes of U_i *intersecting* $\mathbf{D}(0,\rho)$ *satisfies*

$$\#\{\mathrm{D} \in \mathcal{D}(\mathrm{U}_i),\ \mathrm{D} \cap \mathbf{D}(0,\rho) \neq \emptyset\} \lesssim \rho \mathrm{N}_i^2. \tag{10.309}$$

(4) Let $\mathrm{D} \in \mathcal{D}_\rho(\mathrm{U}_{i_+(\rho)})$ *and define*

$$i_{\mathrm{D}} = -1 + \min\{i :\ i_-(\rho) < i \le i_+(\rho),\ \exists \mathrm{D}' \in \mathcal{D}_\rho(\mathrm{U}_{i_+(\rho)}),\ \mathrm{D}' \subset \mathrm{D}\}.$$

Then, D *is of the form* $D = \mathbf{D}(c_D, s_D K_{i_D}^{-1})$, $s_D \in [1,2]$, $c_D \in \mathbf{R}$, $\omega_{i_D}(c_D) \in \{l/k,\ (k,l) \in E_{i_D}\}$ *and one has* $D \subset U_{i_D}$.

(5) Let b_τ *be defined by (6.152) (where* τ *is such that (10.295) is satisfied). One has*

$$\mathbf{D}(0, \rho^{b_\tau}) \subset U_{i_+(\rho)}. \tag{10.310}$$

Proof. — We refer to Appendix I.2 for the proofs of Items 1, 2 and 4.

Proof of Item 3 on the number of holes. — From (10.307) we just have to check that for $N \in \mathbf{N}$

$$\#\{(k,l) \in \mathbf{Z}^2,\ l/k \in]\omega_0 - s, \omega_0 + s[,\ 0 < k < N, 0 \le |l| \le N\} \lesssim sN^2.$$

If (k,l) belongs to the preceding set one has $|l - k\omega_0| < sN$ and thus (k,l) belongs to $[-N,N]^2 \cap \{(x,y) \in \mathbf{R}^2,\ |x - \omega_0 y| \le sN\}$ a set which has Lebesgue measure $\lesssim sN^2$. We thus have for large N, $\#(\mathbf{Z}^2 \cap [-N,N]^2 \cap \{(x,y) \in \mathbf{R}^2,\ |x - \omega_0 y| \le sN\} \lesssim sN^2$.

Proof of Item 5, inclusion (10.310). — Recall that $b_\tau \ge \tau + 1$. Since ω_0 is in $DC(\tau)$, for $(k,l) \in E_j$, $j \le i_+(\rho) - 1$ one has $|l/k - \omega_0| \gtrsim N_j^{-(1+\tau)}$. Since Ω_j satisfies a $(2A, 2B)$-twist condition $(2A)^{-1} \le \partial\omega_j \le 2A$ one has $|c_{l/k}^{(j)}| \gtrsim N_j^{-(1+\tau)}$ and because $K_j^{-1} \ll N_j^{-(1+\tau)}$ (*cf.* (7.163)) one has $|c_{l/k}^{(j)}| - 2K_j^{-1} \gtrsim N_j^{-(1+\tau)} \ge C^{-1} N_{i_+(\rho)}^{-(1+\tau)}$, for some $C > 0$. Now (7.193) shows that $U_{i_+(\rho)}$ contains a disk $\mathbf{D}(0, C^{-1} N_{i_+(\rho)}^{-(\tau+1)})$ and we observe that from (10.305), $(\tau+1)(\mu/\iota(\rho)) < \tau + 1 \le b_\tau$ hence

$$\mathbf{D}(0, \rho^{b_\tau}) \subset \mathbf{D}(0, C^{-1}\rho^{(\tau+1)\mu/\iota(\rho)}) \subset U_{i_+(\rho)}. \tag{10.311}$$

10.1.2. *Covering the holes with bigger disks.* — Let us define (compare with (7.163))

$$\widehat{K}_i = N_i^{\ln N_i} \ll K_i \ll e^{N_i/(\ln N_i)^3} \tag{10.312}$$

and for any $D \in \mathcal{D}_\rho := \mathcal{D}_\rho(U_{i_+(\rho)})$ set

$$\widehat{D} = \mathbf{D}(c_D, \widehat{K}_{i_D}^{-1}), \qquad \widehat{\mathcal{D}}_\rho = \{\widehat{D},\ D \in \mathcal{D}_\rho\}.$$

Notice that for any $a > 0$, $\rho \ll_a 1$ and $i_-(\rho) \le i_D \le i_+(\rho)$ one has

$$\bar{\varepsilon}_{i_D}^{1/a} \ll \widehat{K}_{i_D}^{-1} \ll |c_D|/4. \tag{10.313}$$

Indeed, the inequality of the RHS is due to the fact that $|c_D| > \rho^{b_\tau}$ (*cf.* Proposition 10.2, Item 5) combined with the fact that $N_{i_-(\rho)}^{-1} \lesssim \rho^{1/(1+\tau)}$ (*cf.* (10.306)).

The inequality of the LHS is a consequence of (7.163).

Let us mention that these disks $\widehat{D}$ are the ones on which we shall later perform a Hamilton-Jacobi Normal Form as described in Proposition 8.1.

Lemma **10.3.** — *The elements of* $\widehat{\mathcal{D}}_\rho$ *are pairwise disjoint and for any* $\mathrm{D} \in \mathcal{D}_\rho$ *one has*

$$\mathrm{D} \subset (1/10)\widehat{\mathrm{D}} \subset 6\widehat{\mathrm{D}} \subset \mathrm{U}_{i_{\mathrm{D}}}, \qquad \widehat{\mathrm{D}} \setminus (1/10)\widehat{\mathrm{D}} \subset \mathrm{U}_{i_+(\rho)}.$$

Proof. — Let D and D′ be two distinct elements of $\mathcal{D}_\rho$. By Proposition 10.2, Item 1, $\mathrm{D} \cap \mathrm{D}' = \emptyset$ hence from Lemma 7.3, Item 1 $|c_{\mathrm{D}} - c_{\mathrm{D}'}| \gtrsim \mathrm{N}_{i_+(\rho)}^{-2}$. Since $\widehat{\mathrm{K}}_{i_{\mathrm{D}}}^{-1} + \widehat{\mathrm{K}}_{i_{\mathrm{D}'}}^{-1} \ll \mathrm{N}_{i_+(\rho)}^{-2}$ we get that $\mathbf{D}(c_{\mathrm{D}}, \widehat{\mathrm{K}}_{i_{\mathrm{D}}}^{-1}) \cap \mathbf{D}(c_{\mathrm{D}'}, \widehat{\mathrm{K}}_{i_{\mathrm{D}'}}^{-1}) = \emptyset$.

Let us now prove $6\widehat{\mathrm{D}} \subset \mathrm{U}_{i_{\mathrm{D}}}$. If $6\widehat{\mathrm{D}}$ is not a subset of $\mathrm{U}_{i_{\mathrm{D}}}$ one has for some $\mathrm{D}' \in \mathcal{D}(\mathrm{U}_{i_{\mathrm{D}}})$, $(6\widehat{\mathrm{D}}) \cap \mathrm{D}' \neq \emptyset$ hence $|c_{\mathrm{D}} - c_{\mathrm{D}'}| \leq 6\widehat{\mathrm{K}}_{i_{\mathrm{D}}}^{-1} + \mathrm{K}_{i_{\mathrm{D}'}}^{-1} \ll \mathrm{N}_{i_+(\rho)}^{-2}$. We can apply Lemma 7.3, Item 1 to deduce $|c_{\mathrm{D}} - c_{\mathrm{D}'}| \lesssim \overline{\varepsilon}_{i_-(\rho)}^{1/2}$; but this implies that $\mathrm{D} \cap \mathrm{D}' \neq \emptyset$, hence $\mathrm{D} = \mathrm{D}'$ (we can apply Proposition 10.2, Item 1, since $\mathrm{D}, \mathrm{D}' \in \mathcal{D}_{(3/2)\rho}$) and by Proposition 10.2, Item 4 we obtain $\mathrm{D}' \subset \mathrm{U}_{i_{\mathrm{D}}}$: a contradiction.

Let us prove the second inclusion $\widehat{\mathrm{D}} \setminus (1/10)\widehat{\mathrm{D}} \subset \mathrm{U}_{i_+(\rho)}$. If this is not the case then for some $\mathrm{D}' \in \mathcal{D}(\mathrm{U}_{i_+(\rho)})$ one has $\mathrm{D}' \cap (\widehat{\mathrm{D}} \setminus (1/10)\widehat{\mathrm{D}}) \neq \emptyset$ hence $|c_{\mathrm{D}} - c_{\mathrm{D}'}| \lesssim \mathrm{K}_{i_{\mathrm{D}}}^{-1} \ll \mathrm{N}_{i_+(\rho)}^{-2}$ which implies as before using Lemma 7.3 that $\mathrm{D} = \mathrm{D}'$. But since $\mathrm{D} \subset (1/10)\widehat{\mathrm{D}}$ this leads to a contradiction (otherwise $\mathrm{D}' \cap (\widehat{\mathrm{D}} \setminus (1/10)\widehat{\mathrm{D}}) = \emptyset$). □

Remark **10.1.** — Let us mention (this will be useful in the proof of Theorem 12.3) that

$$\sum_{\widehat{\mathrm{D}} \in \widehat{\mathcal{D}}_\rho} |\widehat{\mathrm{D}} \cap \mathbf{R}|^{1/2} \leq 1.$$

10.1.3. *No-Screening Property.* — Our key proposition is the following.

Proposition **10.4.** — *For any* $\mathrm{D} \in \mathcal{D}(\mathrm{U}_{i_+(\rho)})$ *such that* $\mathrm{D} \cap \mathbf{D}(0, \rho) \neq \emptyset$ *the triple* $(\mathrm{U}_{i_+(\rho)}, \widehat{\mathrm{D}} \setminus (1/10)\widehat{\mathrm{D}}, \mathbf{D}(0, \rho^{b_\tau}/2))$ *is* $(10 b_\tau)^{-1} |\ln \rho|^{-1}$*-good (in the sense of Definition 3.3).*

Proof. — From Remark 3.1 it is enough to prove that for some $\mathrm{U}' \subset \mathrm{U}_{i_+(\rho)}$ containing both $\mathbf{D}(0, \rho^{b_\tau})$ and $\widehat{\mathrm{D}} \setminus (1/10)\widehat{\mathrm{D}}$, the triple $(\mathrm{U}', \widehat{\mathrm{D}} \setminus (1/10)\widehat{\mathrm{D}}, \mathbf{D}(0, \rho^{b_\tau}/2))$ is $(10 b_\tau)^{-1} |\ln \rho|^{-1}$-good.

Lemma **10.5.** — *There exists a constant* $\mathrm{C} > 0$ *such that for any* $1 \leq s \leq 4/3$, *there exists* $\rho' \in [s\rho, s\rho + 10\mathrm{C}\rho^2]$ *such that*

$$\mathbf{D}(0, \rho') \cap \mathrm{U}_{i_+(\rho)} = \mathbf{D}(0, \rho') \setminus \bigcup_{\substack{\mathrm{D} \in \mathcal{D}(\mathrm{U}_{i_+(\rho)}) \\ \mathrm{D} \subset \mathbf{D}(0, \rho')}} \mathrm{D}.$$

Proof. — From Lemma 7.3 the holes of $\mathcal{D}(\mathrm{U}_{i_+(\rho)})$ are $\mathrm{C}_1^{-1} \mathrm{N}_{i_+(\rho)}^{-2}$-separated (some $\mathrm{C}_1 > 0$), hence for some $\mathrm{C}_2 > 0$ they are $\mathrm{C}_2^{-1} \rho^{2\mu/\iota(\rho)}$-separated (*cf.* (10.304)) and because of (10.305) they are $\mathrm{C}^{-1}\rho^2$-separated for some $\mathrm{C} > 0$.

However, each of these disks has a radius $\leq 2K_{i_-(\rho)}^{-1} \ll \rho^4$. Since they are centered on the real line the conclusion follows. □

From the previous lemma we deduce the existence of a $\rho' \in [(5/4)\rho, (4/3)\rho]$ such that all the holes $D \in \mathcal{D}(U_{i_+(\rho)})$ of $U_{i_+(\rho)}$ intersecting $\mathbf{D}(0,\rho')$ are indeed included in $\mathbf{D}(0,\rho')$. We then set

$$U' = U_{i_+(\rho)} \cap \mathbf{D}(0,\rho') = \mathbf{D}(0,\rho') \setminus \bigcup_{\substack{D \in \mathcal{D}(U_{i_+(\rho)}) \\ D \subset \mathbf{D}(0,\rho')}} D.$$

From (10.310) we have $\mathbf{D}(0,\rho^{b_\tau}) \subset U'$ and for any $D = \mathbf{D}(c_D, K_{i_D}^{-1}) \in \mathcal{D}(U_{i_+(\rho)})$ such that $D \cap \mathbf{D}(0,\rho) \neq \emptyset$ one has $\widehat{D} \subset \mathbf{D}(0,(5/6)\rho')$: indeed, since $D \cap \mathbf{D}(0,\rho) \neq \emptyset$, $|c_D| < \rho + K_{i_D}^{-1} < \rho + \rho^4$ hence $|c_D| + \widehat{K}_{i_D}^{-1} < \rho + 2\rho^4 < (5/6)\rho'$. On the other hand, from Lemma 10.3 $\widehat{D} \setminus (1/10)\widehat{D} \subset U'$ ($\widehat{D} \subset \mathbf{D}(0,\rho')$). In this situation we can apply Corollary 3.4 with $U = U'$, $B = \mathbf{D}(0,\rho^{b_\tau}/2)$, $d_i = \widehat{K}_{i_D}^{-1}$, $\varepsilon_i = 2K_{i_D}^{-1}$: the triple $(U', \widehat{D} \setminus (1/10)\widehat{D}, \mathbf{D}(0,\rho^{b_\tau}/2))$ is A-good with

$$A = \frac{\ln(6/5)}{b_\tau |\ln \rho|} - (I) \tag{10.314}$$

where

$$(I) := \sum_{i=i_-(\rho)}^{i_+(\rho)-1} \#\mathcal{C}_i(\rho) \frac{\ln(\widehat{K}_i^{-1}/(20\rho'))}{\ln(2K_i^{-1}/(\rho'))}$$

with

$$\mathcal{C}_i(\rho) = \#\{D \in \mathcal{D}(U_{i_+(\rho)}),\ D \cap \mathbf{D}(0,\rho) \neq \emptyset,\ i_D = i\}.$$

From (10.309) of Proposition 10.2, (10.302), (10.303), (10.312), (7.163) one has

$$\begin{aligned}
(I) &\leq \rho \sum_{i=i_-(\rho)}^{i_+(\rho)-1} N_i^2 \frac{\ln(\widehat{K}_i^{-1} N_{i_-(\rho)}^{\iota(\rho)}/30)}{\ln(2K_i^{-1} N_{i_-(\rho)}^{\iota(\rho)})} \\
&\leq \rho \sum_{i=i_-(\rho)}^{i_+(\rho)-1} N_i^2 \frac{-(\ln N_i)^2 + \ln(N_{i_-(\rho)}^{\iota(\rho)}/30)}{-(1/(2(\overline{a}_0+2)))hN_i/(\ln N_i)^2 + \ln(N_{i_-(\rho)}^{\iota(\rho)}/2)} \\
&\lesssim \rho \sum_{i=i_-(\rho)}^{i_+(\rho)-1} (N_i)^{1+\beta/2} \qquad (\rho \ll_\beta 1)
\end{aligned}$$

and since N_i is exponentially growing with i,

$$(I) \lesssim \rho \times (N_{i_+(\rho)})^{1+\beta/2}.$$

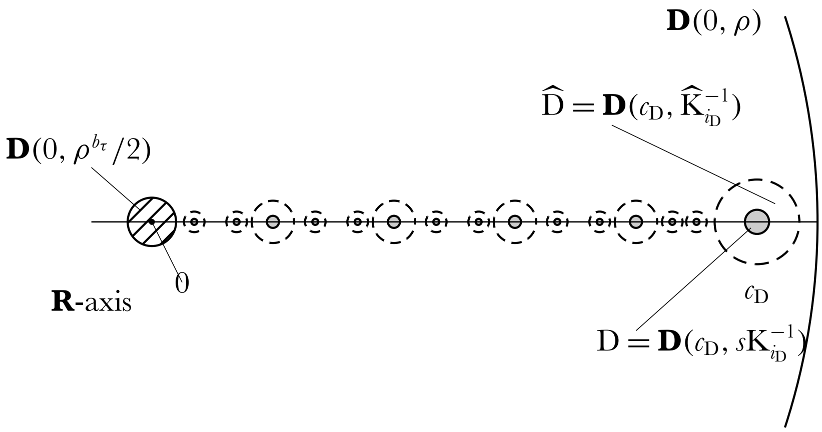

FIG. 9. — Adapted KAM Normal Forms (ω_0 Diophantine) in the complex r-plane. The triple $(\mathrm{U}^{(\rho)}, \widehat{\mathrm{D}} \setminus (1/10)\widehat{\mathrm{D}}, \mathbf{D}(0,\rho^{b_\tau}/2))$ is $\mathrm{C}_b|\ln\rho|^{-1}$-good

From (10.304) and (10.305) we thus get

$$(\mathbf{10.315}) \qquad (\mathrm{I}) \lesssim \rho^{1-(1+\beta/2)\mu/\iota(\rho)} \leq \rho^{\beta^2/4}$$

and from (10.314), if $\rho \ll_\beta 1$

$$\frac{1}{10b_\tau}\frac{1}{|\ln\rho|} \leq \mathrm{A} \qquad (\text{some } \mathrm{C} > 0).$$

□

10.2. *Coexistence of KAM, BNF and HJ Normal Forms on the adapted KAM domain.*

Notation **10.6.** — *If* $\mathrm{W}_{h,\mathrm{U}}$ *is a* σ*-symmetric holed domain, we denote by* $\mathcal{NF}_\sigma(\mathrm{W}_{h,\mathrm{U}})$ *(resp.* $\mathcal{NF}_{ex,\sigma}(\mathrm{W}_{h,\mathrm{U}})$*) the set of triples* (Ω, F, g) *with* $\Omega \in \widetilde{\mathcal{O}}_\sigma(\mathrm{U})$, $\mathrm{F} \in \mathcal{O}_\sigma(\mathrm{W}_{h,\mathrm{U}})$, $g \in \widetilde{\mathrm{Symp}}_\sigma(\mathrm{W}_{h,\mathrm{U}})$ *(resp.* $g \in \widetilde{\mathrm{Symp}}_{ex,\sigma}(\mathrm{W}_{h,\mathrm{U}})$*).*

Proposition **10.7** *(Adapted Normal Forms). — Let* $\Omega \in \widetilde{\mathcal{O}}_\sigma(\mathrm{U})$ *and* $\mathrm{F} \in \mathcal{O}_\sigma(\mathrm{W}_{h,\mathrm{U}})$ *satisfy (10.294), (10.295), (10.296). For any* $\beta \ll_\tau 1$ *define* $i_-(\rho)$, μ *and* $i_+(\rho)$ *according to (10.300), (10.299) and (10.301). Then for any* $\rho \ll_\beta 1$ *the following holds:*

(KAM): *Adapted KAM Normal Form (Proposition 7.1). Let* $\mathrm{D} \in \mathcal{D}_\rho(\mathrm{U}_{i_+(\rho)})$.

$$(\mathbf{10.316}) \qquad [\mathrm{W}_{h,\mathrm{U}_{i_\pm(\rho)}}] \quad g_{1,i_\pm(\rho)}^{-1} \circ \Phi_\Omega \circ f_{\mathrm{F}} \circ g_{1,i_\pm(\rho)} = \Phi_{\Omega_{i_\pm(\rho)}} \circ f_{\mathrm{F}_{i_\pm(\rho)}}$$

$$(\mathbf{10.317}) \qquad [\mathrm{W}_{h,\mathrm{U}_{i_+(\rho)}}] \quad g_{i_{\mathrm{D}},i_+(\rho)}^{-1} \circ \Phi_{\Omega_{i_{\mathrm{D}}}} \circ f_{\mathrm{F}_{i_{\mathrm{D}}}} \circ g_{i_{\mathrm{D}},i_+(\rho)} = \Phi_{\Omega_{i_+(\rho)}} \circ f_{\mathrm{F}_{i_+(\rho)}}$$

$$(\mathbf{10.318}) \qquad [\mathrm{W}_{h,\mathrm{U}_{i_{\mathrm{D}}}}] \quad g_{i_-(\rho),i_{\mathrm{D}}}^{-1} \circ \Phi_{\Omega_{i_-(\rho)}} \circ f_{\mathrm{F}_{i_-(\rho)}} \circ g_{i_-(\rho),i_{\mathrm{D}}} = \Phi_{\Omega_{i_{\mathrm{D}}}} \circ f_{\mathrm{F}_{i_{\mathrm{D}}}}.$$

$$(\mathbf{10.319}) \qquad \|g_{1,i_+(\rho)} - id\|_{\mathrm{C}^1} \lesssim \overline{\varepsilon}^{1/2} \leq \rho^{m/2}$$

$$\|g_{i_D,i_+(\rho)} - id\|_{C^1} \le \overline{\varepsilon}_{i_D}^{1/2} \tag{10.320}$$

$$\|F_{i_+(\rho)}\|_{W_{h,U(\rho)}} \lesssim \exp(-(1/\rho)^{(1/\tau)-2\beta}). \tag{10.321}$$

Note that $(\Omega_i, F_i, g_i) \in \mathcal{NF}_{ex,\sigma}(W_{h,U_i})$ *and* $\Omega_i \in \mathcal{TC}(2A, 2B)$.

(HJ): *Hamilton-Jacobi Normal Form. (Proposition 8.1). For any* $D \in \mathcal{D}_\rho(U_{i_+(\rho)})$ *there exists* $\check{D} \subset \widehat{D}$ *and* $(\Omega^{HJ}_{\widehat{D}}, F^{HJ}_{\widehat{D}}, g^{HJ}_{\widehat{D}}) \in \mathcal{NF}_\sigma(W_{h/9,\widehat{D}\setminus\check{D}})$ *such that*

$$(g^{HJ}_{\widehat{D}})^{-1} \circ \Phi_{\Omega_{i_D}} \circ f_{F_{i_D}} \circ g^{HJ}_{\widehat{D}} = \Phi_{\Omega^{HJ}_{\widehat{D}}} \circ f_{F^{HJ}_{\widehat{D}}} \qquad [W_{h/9,\widehat{D}\setminus\check{D}}] \tag{10.322}$$

$$\|g^{HJ}_{\widehat{D}} - id\|_{C^1} \lesssim \overline{\varepsilon}_{i_D}^{1/9} \tag{10.323}$$

$$\Omega^{HJ}_{\widehat{D}} \in \mathcal{TC}(2A, 2A) \tag{10.324}$$

$$\|F^{HJ}_{\widehat{D}}\|_{W_{h/9,(\widehat{D}\setminus\check{D})}} \lesssim \exp(-(1/\rho)). \tag{10.325}$$

The triple $(\Omega^{HJ}_D, \widehat{D}, \check{D})$ *satisfies the Extension Principle of Proposition 8.1.*

(BNF): *Birkhoff Normal Form (Proposition 6.5):*
There exists $(\Omega^{BNF}_\rho, F^{BNF}_\rho, g^{BNF}_\rho) \in \mathcal{NF}_{ex,\sigma}(W_{h,\mathbf{D}(0,\rho^{b_\tau})})$ *such that*

$$(g^{BNF}_\rho)^{-1} \circ \Phi_\Omega \circ f_F \circ g^{BNF}_\rho = \Phi_{\Omega^{BNF}_\rho} \circ f_{F^{BNF}_\rho}, \qquad (W_{h,\mathbf{D}(0,\rho^{b_\tau})}) \tag{10.326}$$

$$\|g^{BNF}_\rho - id\|_{C^1} \lesssim \rho^{m-10}. \tag{10.327}$$

$$\Omega^{BNF}_\rho \in \mathcal{TC}(2A, 2B) \tag{10.328}$$

$$\|F^{BNF}_\rho\|_{W_{h,\mathbf{D}(0,\rho^{b_\tau})}} \lesssim \exp(-(1/\rho)^{1-\beta}) \tag{10.329}$$

Proof. — **KAM:** This is just the content of Proposition 7.1. For inequality (10.321) we note that from (7.170), (7.163), (10.304), (10.305)

$$\begin{aligned}\|F_{i_+(\rho)}\|_{h,U_{i_+(\rho)}} &\lesssim \exp(-N_{i_+(\rho)}/(\ln(N_{i_+(\rho)}))^2)\\ &\lesssim \exp(-\rho^{-(\mu/\iota(\rho))^-})\\ &\lesssim \exp(-(1/\rho)^{(1/\tau)-2\beta}).\end{aligned}$$

HJ: Let $D \in \mathcal{D}_\rho(U_{i_+(\rho)})$ where $D = \mathbf{D}(c_D, s_D K_{i_D}^{-1})$, $\omega_{i_D}(c_D) = p/q$, $q \le N_{i_D}$, $p \wedge q = 1$, be one of the disks obtained in Proposition 10.2, Item 4. By Lemma 10.3 the disk $6\widehat{D} = \mathbf{D}(c_D, 6\widehat{K}_{i_D}^{-1})$ is included in U_{i_D}. We observe that $6\widehat{K}_{i_D}^{-1} < |c_D|/4$ (*cf.* (10.313)). Since

$$\min(6\widehat{K}_{i_D}^{-1}, |c_D|/4) = 6\widehat{K}_{i_D}^{-1} < (Aq)^{-8} \quad \text{and} \quad \|F_{i_D}\|_{h,6\widehat{D}} \lesssim \overline{\varepsilon}_{i_D} < (6\widehat{K}_{i_D}^{-1})^{\overline{a}_3}$$

(the last inequality comes also from (10.313)) condition (8.205), (8.203) are satisfied and we can apply Proposition 8.1 on Hamilton-Jacobi Normal Forms to $\Phi_{\Omega_{i_D}}\circ f_{F_{i_D}}$ on the domain $W_{h,\widehat{D}}\subset W_{h,U_{i_D}}$ with $\widehat{\rho}=\widehat{K}_{i_D}^{-1}$: there exists a disk $\check{D}\subset\widehat{D}$

$$\check{D}:=\mathbf{D}(c_{\check{D}},\rho_{\check{D}})\subset(1/10)\widehat{D}:=\mathbf{D}(c_D,(1/10)\widehat{K}_{i_D}^{-1})\subset U_{i_D} \tag{10.330}$$

and $(\Omega_{\widehat{D}}^{HJ},F_{\widehat{D}}^{HJ},g_{\widehat{D}}^{HJ})\in\mathcal{NF}_\sigma(W_{h/9,\widehat{D}\setminus\check{D}})$ satisfying (10.322)

$$\|g_{\widehat{D}}^{HJ}-id\|_{C^1}\lesssim q\bar{\varepsilon}_{i_D}^{1/8}\leq\bar{\varepsilon}_{i_D}^{1/9} \tag{10.331}$$

$$\|F_{\widehat{D}}^{HJ}\|_{W_{h/9,(\widehat{D}\setminus\check{D})}}\lesssim\exp(-(\widehat{K}_{i_D}/N_{i_D})^{1/4}). \tag{10.332}$$

To obtain inequality (10.325) we observe that since $\widehat{K}_{i_D}=N_{i_D}^{\ln N_{i_D}}$ with $i_+(\rho)\geq i_D\geq i_-(\rho)$ we get

$$-(\widehat{K}_{i_D}/N_{i_D})\lesssim -N_{i_-(\rho)}^{-1+\ln N_{i_-(\rho)}}.$$

Because for ρ small enough $-1+\ln N_{i_-(\rho)}\geq 4(2+\tau)$ we get

$$-(\widehat{K}_{i_D}/N_{i_D})^{1/4}\lesssim -N_{i_-(\rho)}^{2+\tau}$$

which yields, using (10.302) and (10.303) ($\rho\ll_\tau 1$)

$$\begin{aligned}-(\widehat{K}_{i_D}/N_{i_D})^{1/4}&<-(1/\rho)^{(2+\tau)/\iota(\rho)}\\&<-(1/\rho).\end{aligned}$$

BNF: We observe that $\mathbf{D}(0,\rho^{b_\tau})\subset\mathbf{D}(0,\rho)$ and apply Proposition 6.5 to (Ω,F) on $e^hW_{h,\mathbf{D}(0,\rho)}$ (we use the smallness condition (10.296)). □

10.3. *Comparision Principle.* — We now use the result of Section 9 to show that these various Normal Forms match to some very good order of approximation.

Lemma **10.8** *(Comparing Adapted Normal Forms). — For any* $\beta\ll_\tau 1$, *and* $\rho\ll_\beta 1$

$$\|\Omega_{i_+(\rho)}-\Omega_\rho^{BNF}\|_{(1/2)\mathbf{D}(0,\rho^{b_\tau})}\leq\exp(-(1/\rho)^{(1/\tau)-3\beta}) \tag{10.333}$$

and for any $D\in\mathcal{D}_\rho$ *there exists* $\gamma_D\leq\widehat{K}_{i_D}^{-2}$

$$\|\Omega_{i_+(\rho)}-\Omega_{\widehat{D}}^{HJ}(\cdot+\gamma_D)\|_{(4/5)\widehat{D}\setminus(1/5)\widehat{D}}\leq\exp(-(1/\rho)^{(1/\tau)-3\beta}). \tag{10.334}$$

Proof. — 1) *Proof of (10.333).* From (10.326), (10.316) and the fact that

$$W_{h,\mathbf{D}(0,\rho^{b_\tau})}\subset W_{h,\mathbf{D}(0,\rho^{b_\tau})}\cap W_{h,U_{i_+(\rho)}}$$

one has on $g_{1,i_+(\rho)}(W_{h,\mathbf{D}(0,\rho^{b_\tau})})\cap g_\rho^{\rm BNF}(W_{h,\mathbf{D}(0,\rho^{b_\tau})})$

$$g_{1,i_+(\rho)}\circ\Phi_{\Omega_{i_+(\rho)}}\circ f_{F_{i_+(\rho)}}\circ(g_{1,i_+(\rho)})^{-1}=g_\rho^{\rm BNF}\circ\Phi_{\Omega_\rho^{\rm BNF}}\circ f_{F_\rho^{\rm BNF}}\circ(g_\rho^{\rm BNF})^{-1}.$$

We can then apply Propositions 9.1–9.2 with $\rho_2=\rho^{b_\tau}$, $\rho_1=0$, $\delta=\rho^{b_\tau}/2$, $\varepsilon=\rho^{\min(m/2,m-10)}$, $\nu=\exp(-(1/\rho)^{(1/\tau)-2\beta})$ because from (10.329), (10.327), (10.319), (10.321) one sees that condition (9.283) reads

$$\overline{C}\rho^{\min(m/2,m-10)}\le\rho^{b_\tau}/4<\rho^{b_\tau}\quad\text{and}\quad\overline{C}(\rho^{b_\tau}/2)^{-\overline{a}_4}\exp-(1/\rho)^{(1/\tau)-2\beta})<1$$

and is satisfied for $\rho\ll1$ (*cf.* (10.297)). Since $g_{1,i_+(\rho)}$ and $g_\rho^{\rm BNF}$ are exact symplectic we then get $\|\Omega_{i_+(\rho)}-\Omega_\rho^{\rm BNF}\|_{\mathbf{D}(0,(1/2)\rho^{b_\tau})}\le\overline{C}\rho^{-(b_\tau+1)\overline{a}_5}\exp(-(1/\rho)^{(1/\tau)-2\beta})$ which is $\le\exp(-(1/\rho)^{(1/\tau)-3\beta})$ if ρ is small enough.

2) *Proof of (10.334)*. Similarly, from (10.317), (10.322) one has on the set

$$g_{\widehat{D}}^{\rm HJ}(W_{h/9,\widehat{D}\setminus(1/5)\widehat{D}})\cap g_{i_D,i_+(\rho)}(W_{h/9,\widehat{D}\setminus(1/5)\widehat{D}})$$

$$g_{\widehat{D}}^{\rm HJ}\circ\Phi_{\Omega_{\widehat{D}}^{\rm HJ}}\circ f_{F_{\widehat{D}}^{\rm HJ}}\circ(g_{\widehat{D}}^{\rm HJ})^{-1}=g_{i_D,i_+(\rho)}\circ\Phi_{\Omega_{i_+(\rho)}}\circ f_{F_{i_+(\rho)}}\circ(g_{i_D,i_+(\rho)})^{-1}$$

and from (10.320) (10.321), (10.323), (10.325), we see that Propositions 9.1–9.2 apply with $c=c_{i_D}$, $\varepsilon=\overline{\varepsilon}_{i_D}^{1/9}$, $\delta=K_{i_D}^{-1}/20$, $\rho_1=(1/10)\widehat{K}_{i_D}^{-1}$, $\rho_2=K_{i_D}^{-1}<|c_{i_D}|/4$ since condition (9.283) is implied by

$$\overline{C}\overline{\varepsilon}_{i_D}^{1/9}\le\widehat{K}_{i_D}^{-1}/80<\widehat{K}_{i_D}^{-1}/10\quad\text{and}\quad\overline{C}(20\widehat{K}_{i_D})^{\overline{a}_4}\exp(-(1/\rho)^{(1/\tau)-2\beta})<1$$

which is satisfied (*cf.* (7.163), (10.312)) if ρ is small enough. We then get for some $\gamma_D\in\mathbf{R}$, $|\gamma_D|\lesssim\overline{C}\overline{\varepsilon}_{i_D}^{1/9}\le\widehat{K}_{i_D}^{-2}$ (*cf.* (10.313)) that on the annulus $(4/5)\widehat{D}\setminus(1/5)\widehat{D}$ one has $|\Omega_{i_+(\rho)}-\Omega_{\widehat{D}}^{\rm HJ}(\cdot+\gamma_D)|\le\exp(-(1/\rho)^{(1/\tau)-3\beta})$. □

11. Adapted Normal Forms: ω_0 Liouvillian (CC case)

Let $h>0$, $0<\overline{\rho}<1$, $A,B\ge1$, $\Omega\in\widetilde{\mathcal{O}}_\sigma(e^{10h}\mathbf{D}(0,\overline{\rho}))$, $F\in\mathcal{O}_\sigma(e^{10h}W_{h,\mathbf{D}(0,\overline{\rho})})$ such that

(**11.335**) $\quad\forall\, r\in\mathbf{R},\ A^{-1}\le(2\pi)^{-1}\partial^2\Omega(r)\le A,\quad$ and $\ \|(2\pi)^{-1}D^3\Omega\|_{\mathbf{C}}\le B$

(**11.336**) $\quad\omega_0:=(2\pi)^{-1}\partial\Omega(0)\in\mathbf{R}\setminus\mathbf{Q}$

(**11.337**) $\quad\forall\,0<\rho\le\overline{\rho},\quad\|F\|_{e^{10h}W_{h,\mathbf{D}(0,\rho)}}\le\rho^m,$

where

(**11.338**) $\quad m=4+\max(\overline{a}_1,2000A\overline{a}_2,\overline{a}_3,\overline{a}_4)$

($\overline{a}_1,\overline{a}_2,\overline{a}_3,\overline{a}_4$ are the constants appearing in Propositions 6.4, 7.1, 8.1, 9.1).

Using the notations of Section 6.3.1, let $(p_n/q_n)_n$ be the sequence of convergents of ω_0:

$$\frac{1}{2q_{n+1}q_n} \leq |\omega_0 - (p_n/q_n)| \leq \frac{1}{q_{n+1}q_n}. \tag{11.339}$$

$$\forall\, 0 < k < q_n,\ \forall\, l \in \mathbf{Z},\ |\omega_0 - (l/k)| > \frac{1}{2kq_n}. \tag{11.340}$$

We assume that n is large enough and we set

$$\rho_n = \frac{10\mathrm{A}}{q_{n+1}q_n} \leq \overline{\rho}/10. \tag{11.341}$$

We introduce

$$\overline{\varepsilon} := \max_{0\leq j\leq 3} \|\mathrm{D}^j\mathrm{F}\|_{\mathrm{W}_{2h,\mathbf{D}(0,10\rho_n)}} \lesssim (10\rho_n)^{m-3} \leq \rho_n^{2000\mathrm{A}\overline{a}_2}. \tag{11.342}$$

11.1. *Adapted KAM domains.* — Since Condition (7.161) is satisfied we can apply Proposition 7.1 (with $\overline{\rho}_{\text{Prop. 7.1}} = 10\rho_n$) and define holed domains U_i, functions Ω_i, F_i, ω_i *etc.* In particular for $0 < t$

$$\begin{cases} \mathrm{U}_i \cap \mathbf{D}(0,t) = \mathbf{D}(0,t) \setminus \bigcup_{j=1}^{i-1} \bigcup_{(k,l)\in \mathrm{E}_j} \mathbf{D}(c_{l/k}^{(j)}, s_{j,i-1}\mathrm{K}_j^{-1}), \\ s_{j,i-1} = e^{\sum_{m=j}^{i-1}\delta_m} \in [1,2] \end{cases} \tag{11.343}$$

where

$$\mathrm{E}_j \subset \{(k,l) \in \mathbf{Z}^2,\ 0 < k < \mathrm{N}_j,\ 0 \leq |l| \leq \mathrm{N}_j\}, \qquad \omega_j(c_{l/k}^{(j)}) = l/k.$$

Note that from (11.342) and the definition (7.163) of K_j

$$\mathrm{K}_j^{-1} \leq \overline{\varepsilon}^{\frac{1}{2(\overline{a}_0+2)}} \leq \rho_n^{1000\mathrm{A}}. \tag{11.344}$$

Lemma **11.1.** — *Let j be such that $\mathrm{N}_j < q_{n+1}/(10\mathrm{A})^2$ and $(k,l) \in \mathrm{E}_j$.*

(1) If $(k,l) \in \mathbf{Z}(q_n,p_n)$ one has $l/k = p_n/q_n$ and

$$(40\mathrm{A}^2)^{-1}\rho_n \leq \frac{(2\mathrm{A})^{-1}}{2q_{n+1}q_n} \leq |c_{p_n/q_n}^{(j)}| \leq \frac{(2\mathrm{A})}{q_{n+1}q_n} \leq \rho_n/5. \tag{11.345}$$

(2) If $(k,l) \notin \mathbf{Z}(q_n,p_n)$

$$|c_{l/k}^{(j)}| \geq 4\rho_n. \tag{11.346}$$

Proof. — Item 1 comes from (11.339) and the twist condition (7.166).

To prove Item 2 we observe that if $(k,l)\notin \mathbf{Z}(q_n,p_n)$

$$|\omega_0-\frac{l}{k}|\geq |\frac{l}{k}-\frac{p_n}{q_n}|-|\omega_0-\frac{p_n}{q_n}|\geq \frac{1}{kq_n}-\frac{1}{q_nq_{n+1}}\geq \frac{99\mathrm{A}^2}{q_nq_{n+1}}\geq 9\mathrm{A}\rho_n$$

and from the twist condition (7.166) we get $|c^{(j)}_{l/k}|\geq 4\rho_n$. □

For $n\in\mathbf{N}^*$ define i_n^- as the unique index i such that

$$\mathrm{N}_{i_n^- -1}\leq q_n<\mathrm{N}_{i_n^-} \tag{11.347}$$

and i_n^+ as the unique index (see the definition of the sequence N_i in (7.163)) such that

$$\frac{(3/4)q_{n+1}}{(10\mathrm{A})^2}\leq \mathrm{N}_{i_n^+}<\frac{q_{n+1}}{(10\mathrm{A})^2}. \tag{11.348}$$

We define

$$c_n=c^{(i_n^-)}_{p_n/q_n},\qquad \mathrm{D}_n:=\mathbf{D}(c^{(i_n^-)}_{p_n/q_n},s_{i_n^-,i_n^+-1}\mathrm{K}^{-1}_{i_n^-}),\qquad \widehat{\mathrm{D}}_n=\mathbf{D}(c_n,|c_n|/24)$$

$$\mathrm{U}^{(n)}:=\mathrm{U}_{i_n^+}\cap \mathbf{D}(0,\rho_n). \tag{11.349}$$

Note that from (11.345)

$$(40\mathrm{A}^2)^{-1}\rho_n\leq |c_n|\leq \rho_n/5. \tag{11.350}$$

Proposition **11.2.** — *For n large enough,*

(1) $\mathbf{D}(0,\rho_n)\subset \mathrm{U}_{i_n^-}$.
(2) One has $\mathcal{D}_{\rho_n}:=\mathcal{D}(\mathrm{U}^{(n)})=\{\mathrm{D}_n\}$.
(3) One has the following inclusion $6\widehat{\mathrm{D}}_n\subset \mathrm{U}_{i_n^-}$.
(4) One has $\mathbf{D}(0,q_{n+1}^{-6})\subset \mathrm{U}_{i_n^+}$.
(5) The triple $(\mathrm{U}^{(n)},\widehat{\mathrm{D}}_n\setminus(1/10)\widehat{\mathrm{D}}_n,\mathbf{D}(0,q_{n+1}^{-6}/2))$ *is* $1/(10|\ln\rho_n|)$*-good (in the sense of Definition 3.3).*

Proof of Item 1. — If $j<i_n^-$ and $(k,l)\in \mathrm{E}_j$ one has $0<k<\mathrm{N}_{i_n^- -1}\leq q_n$ hence from (11.346) $|c^{(j)}_{l/k}|\geq 4\rho_n$ and from (11.344) $|c^{(j)}_{l/k}|-2\mathrm{K}_j^{-1}\geq 3\rho_n$. The conclusion then follows from (11.343) applied with $i=i_n^-$.

Proof of Item 2. — From Item 1, equality (11.343) can be written

$$\mathrm{U}_{i_n^+}\cap\mathbf{D}(0,\rho_n)=\mathbf{D}(0,\rho_n)\setminus\bigcup_{j=i_n^-}^{i_n^+-1}\bigcup_{(k,l)\in \mathrm{E}_j}\mathbf{D}(c^{(j)}_{l/k},s_{j,i-1}\mathrm{K}_j^{-1}).$$

We observe that $(q_n, p_n) \in E_{i_n^-}$ and from (11.345), (11.344) one sees that $D_n = \mathbf{D}(c^{(i_n^-)}_{p_n/q_n}, s_{i_n^-, i_n^+ - 1} K^{-1}_{i_n^-}) \subset \mathbf{D}(0, \rho_n)$. More generally, if $(k, l) \in E_j$, $i_n^- \leq j \leq i_n^+ - 1$ and $(k, l) \notin \mathbf{Z}(q_n, p_n)$, one has $N_j \leq q_{n+1}/(10A)^2$ and (11.346), (11.344) give that $\mathbf{D}(0, c^{(j)}_{l/k}, 2K_j^{-1}) \cap \mathbf{D}(0, \rho_n) = \emptyset$. Since the sets $\mathbf{D}(0, c^{(j)}_{p_n/q_n}, s_{q_n, i^n_+ - 1} K_j^{-1})$, $i_n^- \leq j \leq i_n^+$, form a nested decreasing (for the inclusion) sequence of disks one gets

$$U_{i_n^+} \cap \mathbf{D}(0, \rho_n) = \mathbf{D}(0, \rho_n) \setminus D_n.$$

Proof of Item 3. — This comes from the fact that $|c_n| + 6|c_n|/4 \leq \rho_n$.

Proof of Item 4. — This comes from Item 1 and the fact that $|c_n| - |c_n|/4 \geq q_{n+1}^{-6}$ as is clear from the LHS inequality of (11.345).

Proof of Item 5. — Notice that from (11.345) $5 \leq \rho_n/|c_n| \leq 40A^2$ and that $2K^{-1}_{i_n^-} \leq \rho_n^{1000A}$. We use Corollary 3.4; we have to evaluate

$$\begin{aligned} I &= \frac{\ln(|c_n|/(4\rho_n))}{\ln(q_{n+1}^{-6}/(2\rho_n))} - \frac{\ln(|c_n|/8\rho_n)}{\ln(2K^{-1}_{i_n^-}/\rho_n)} \\ &\geq \frac{\ln(20)}{7|\ln \rho_n|} - \frac{\ln(320A^2)}{(1000A - 1)|\ln \rho_n|} \\ &\geq \frac{1}{10|\ln \rho_n|}. \end{aligned}$$ □

11.2. *Adapted Normal Forms.*

Proposition **11.3.** — *Let* $\Omega \in \widetilde{\mathcal{O}}_\sigma(U)$ *and* $F \in \mathcal{O}_\sigma(W_{h,U})$ *satisfy (10.294), (10.295), (10.296). Let* $0 < \beta \ll 1$ *and* $n \gg_\beta 1$ *such that*

$$q_{n+1} \geq q_n^{10}. \tag{11.351}$$

(KAM): *Adapted KAM Normal Form ((Proposition 7.1)): One has* $(\Omega_i, F_i, g_i \in \mathcal{NF}_{ex,\sigma}(W_{h,U_i}),$ $\Omega_i \in \mathcal{TC}(2A, 2B)$ *and*

$$g^{-1}_{1,i_n^\pm} \circ \Phi_\Omega \circ f_F \circ g_{1,i_n^\pm} = \Phi_{\Omega_{i_n^\pm}} \circ f_{F_{i_n^\pm}} \qquad [W_{h,U_{i_n^\pm}}] \tag{11.352}$$

$$g^{-1}_{i_n^-, i_n^+} \circ \Phi_{\Omega_{i_n^-}} \circ f_{F_{i_n^-}} \circ g_{i_n^-, i_n^+} = \Phi_{\Omega_{i_n^+}} \circ f_{F_{i_n^+}} \qquad [W_{h,U^{(n)}}] \tag{11.353}$$

$$\|g_{1,i_n^+} - id\|_{C^1}, \|g_{i_n^-, i_n^+} - id\|_{C^1} \lesssim \overline{\varepsilon}^{1/2} \leq \rho_n^{m/3} \tag{11.354}$$

$$\|F_{i_n^+}\|_{W_{h,U^{(n)}}} \lesssim \exp(-q_{n+1}^{1-\beta}). \tag{11.355}$$

(HJ): *Hamilton-Jacobi Normal Form (Proposition 8.1).*

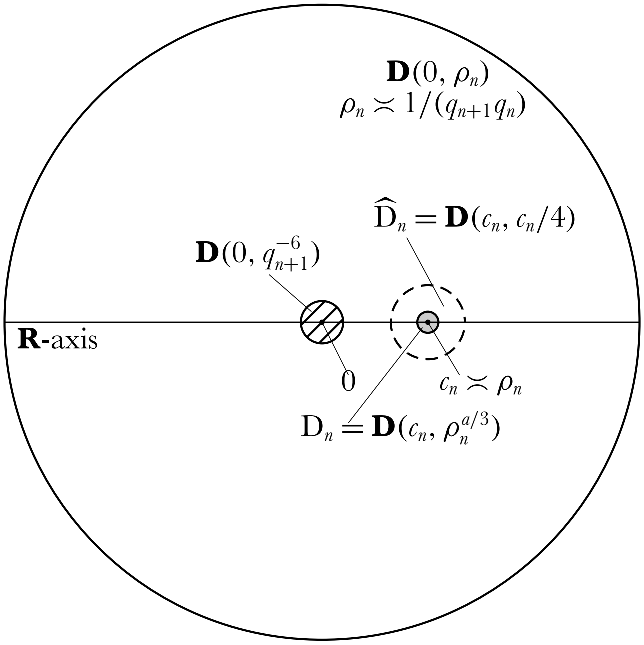

FIG. 10. — Adapted KAM Normal Forms (CC Case) in the complex r-plane. The triple $(\mathrm{U}^n, \widehat{\mathrm{U}}^n, \mathbf{D}(0, q_{n+1}^{-6}))$ is $1/(10|\ln \rho_n|)$-good

There exists $(\Omega_n^{\mathrm{HJ}}, \mathrm{F}_n^{\mathrm{HJ}}, g_n^{\mathrm{HJ}}) \in \mathcal{NF}_\sigma(\mathrm{W}_{h/9, \widehat{\mathrm{D}}_n \smallsetminus \check{\mathrm{D}}_n})$ *such that*

$$(\mathbf{11.356}) \qquad (g_n^{\mathrm{HJ}})^{-1} \circ \Phi_{\Omega_{i_n^-}} \circ f_{\mathrm{F}_{i_n^-}} \circ g_n^{\mathrm{HJ}} = \Phi_{\Omega_n^{\mathrm{HJ}}} \circ f_{\mathrm{F}_n^{\mathrm{HJ}}} \qquad [\mathrm{W}_{h/9,(\widehat{\mathrm{D}}_n \smallsetminus \check{\mathrm{D}}_n)}]$$

$$(\mathbf{11.357}) \qquad \|g_n^{\mathrm{HJ}} - id\|_{\mathrm{C}^1} \lesssim \overline{\varepsilon}_{i_n^-}^{1/9} \leq \rho_n^{m/9}$$

$$(\mathbf{11.358}) \qquad \Omega_n^{\mathrm{HJ}} \in \mathcal{TC}(2\mathrm{A}, 2\mathrm{A})$$

$$(\mathbf{11.359}) \qquad \|\mathrm{F}_n^{\mathrm{HJ}}\|_{\mathrm{W}_{h/9,(\widehat{\mathrm{D}}_n \smallsetminus \check{\mathrm{D}}_n)}} \lesssim \exp(-q_{n+1}^{(1/4)-\beta}).$$

The triple $(\Omega_{\mathrm{D}}^{\mathrm{HJ}}, \widehat{\mathrm{D}}, \check{\mathrm{D}})$ *satisfies the Extension Principle of Proposition* 8.1.

(BNF): *Birkhoff Normal (Proposition* 6.4*):*
There exists $(\Omega_{q_{n+1}^{-1}}^{\mathrm{BNF}}, \mathrm{F}_{q_{n+1}^{-1}}^{\mathrm{BNF}}, g_{q_{n+1}^{-1}}^{\mathrm{BNF}}) \in \mathcal{NF}_{ex,\sigma}(\mathrm{W}_{h,\mathbf{D}(0,q_{n+1}^{-6})})$ *such that*

$$(\mathbf{11.360}) \qquad (g_{q_{n+1}^{-1}}^{\mathrm{BNF}})^{-1} \circ \Phi_\Omega \circ f_{\mathrm{F}} \circ g_{q_{n+1}^{-1}}^{\mathrm{BNF}} = \Phi_{\Omega_{q_{n+1}^{-1}}^{\mathrm{BNF}}} \circ f_{\mathrm{F}_{q_{n+1}^{-1}}^{\mathrm{BNF}}} \qquad [\mathrm{W}_{h,\mathbf{D}(0,q_{n+1}^{-6})}]$$

$$(\mathbf{11.361}) \qquad \|g_{q_{n+1}^{-1}}^{\mathrm{BNF}} - id\|_{\mathrm{W}_{h,\mathbf{D}(0,q_{n+1}^{-6})}} \lesssim q_{n+1}^{-(m-27)}$$

$$(\mathbf{11.362}) \qquad \Omega_{q_{n+1}^{-1}}^{\mathrm{BNF}} \in \mathcal{TC}(2\mathrm{A}, 2\mathrm{B})$$

$$(\mathbf{11.363}) \qquad \|\mathrm{F}_{q_{n+1}^{-1}}^{\mathrm{BNF}}\|_{\mathrm{W}_{h,\mathbf{D}(0,q_{n+1}^{-6})}} \leq \exp(-q_{n+1}^{1-\beta}).$$

Proof. — **KAM:** This is Proposition 7.1. Inequality (11.355) comes from the corresponding (7.170) $\overline{\varepsilon}_{i_n^+} \le \exp(-N_{i_n^+}/(\ln N_{i_n^+})^3)$ and the fact that $N_{i_n^+} \asymp q_{n+1}$.

HJ: By Proposition 11.2, Item 3, the disk $6\widehat{D}_n = \mathbf{D}(c_n, |c_n|/4)$ is included in $U_{i_n^-}$. Since

$$(6(|c_n|/24))^{1/8} < (Aq_n)^{-1}, \quad \text{and} \quad \|F_{i_n^-}\|_{h,6\widehat{D}_n} \lesssim \overline{\varepsilon}_{i_n^-} < (|c_n|/4)^{\overline{a}_3}$$

(the first inequality is a consequence of (11.351) and the second of (11.342) and the fact that $|c_n| \asymp \rho_n$) (8.205), (8.203) are satisfied and we can apply Proposition 8.1 on Hamilton-Jacobi Normal Forms to $\Phi_{\Omega_{i_n^-}} \circ f_{F_{i_n^-}}$ on the domain $W_{h,\widehat{D}_n} \subset W_{h,U_{i_n^-}}$ with $\widehat{\rho} = |c_n|/24$: there exists a disk $\check{D}_n \subset \widehat{D}_n$

$$\check{D}_n := \mathbf{D}(c_{\check{D}_n}, \rho_{\check{D}_n}) \subset (1/10)\widehat{D}_n \subset \widehat{D}_n = \mathbf{D}(c_n, |c_n|/24) \subset U_{i_n^-} \tag{11.364}$$

and $(\Omega_n^{HJ}, F_n^{HJ}, g_n^{HJ}) \in \mathcal{NF}_\sigma(W_{h/9,\widehat{D}_n \smallsetminus \check{D}_n})$ such that one has (11.356) and

$$\|g_{\widehat{D}_n}^{HJ} - id\|_{W_{h/9,(\widehat{D}_n \smallsetminus \check{D}_n)}} \lesssim q_n \overline{\varepsilon}_{i_n^-}^{1/8} \le \overline{\varepsilon}_{i_n^-}^{1/9} \tag{11.365}$$

$$\|F_{\widehat{D}_n}^{HJ}\|_{W_{h/9,(\widehat{D}_n \smallsetminus \check{D}_n)}} \lesssim \exp(-1/(q_n|c_n|/24)^{1/4}) \tag{11.366}$$

and since $|c_n| \asymp (q_n q_{n+1})^{-1}$ $(n \gg_\beta 1)$

$$\|F_{\widehat{D}_n}^{HJ}\|_{W_{h/40,(\widehat{D}_n \smallsetminus \check{D}_n)}} \lesssim \exp(-q_{n+1}^{(1/4)-\beta}).$$

BNF: Since $\|F\|_{e^{1/10}W_{h,\mathbf{D}(0,\rho_n)}} \le \rho_n^{\overline{a}_1}$ (*cf.* (11.337)) we can apply Proposition 6.4 on the existence of approximate BNF in the CC case (with $n+1$ in place of n): for $0 < \beta \ll 1$ and $n \gg_\beta 1$: there exists $(\Omega^{BNF}_{q_{n+1}^{-1}}, F^{BNF}_{q_{n+1}^{-1}}, g^{BNF}_{q_{n+1}^{-1}}) \in \mathcal{NF}_{ex,\sigma}(W_{h,\mathbf{D}(0,q_{n+1}^{-6})})$ such that

$$[W_{h,\mathbf{D}(0,q_{n+1}^{-6})}] \qquad (g^{BNF}_{q_{n+1}^{-1}})^{-1} \circ \Phi_\Omega \circ f_F \circ g^{BNF}_{q_{n+1}^{-1}} = \Phi_{\Omega^{BNF}_{q_{n+1}^{-1}}} \circ f_{F^{BNF}_{q_{n+1}^{-1}}}$$

with

$$\|F^{BNF}_{q_{n+1}^{-1}}\|_{W_{h,q_{n+1}^{-6}}} \le \exp(-q_{n+1}^{1-\beta}). \qquad \square$$

11.3. *Comparision Principle.* — These various Normal Forms match to some very good order of approximation.

Lemma **11.4** *(Comparing Adapted Normal Forms). — One has for any $\beta \ll 1$, $n \gg_\beta 1$*

$$\|\Omega_{i_n^+} - \Omega^{BNF}_{q_{n+1}^{-1}}\|_{(1/2)\mathbf{D}(0,q_{n+1}^{-6})} \lesssim \exp(-q_{n+1}^{1-\beta}) \tag{11.367}$$

and there exists $\gamma_n \lesssim q_{n+1}^{-m} \lesssim |c_n|^{m/2}$ such that

$$\|\Omega_{i_n^+} - \Omega^{HJ}_{\widehat{D}_n}(\cdot + \gamma_n)\|_{(4/5)\widehat{D}_n \smallsetminus (1/5)\widehat{D}_n} \lesssim \exp(-q_{n+1}^{(1/4)-\beta}) \tag{11.368}$$

Proof. — Let us prove estimate (11.367). From (11.360)–(11.352), we see that on $g_{1,i_n^+}(W_{h,\mathbf{D}(0,q_{n+1}^{-6})}) \cap g^{\mathrm{BNF}}_{q_{n+1}^{-1}}(W_{h,\mathbf{D}(0,q_{n+1}^{-6})})$ one has

$$g^{\mathrm{BNF}}_{q_{n+1}^{-1}} \circ \Phi_{\Omega^{\mathrm{BNF}}_{q_{n+1}^{-1}}} \circ f_{F^{\mathrm{BNF}}_{q_{n+1}^{-1}}} \circ (g^{\mathrm{BNF}}_{q_{n+1}^{-1}})^{-1} = g_{1,i_n^+} \circ \Phi_{\Omega_{i_n^+}} \circ f_{F_{i_n^+}} \circ (g_{1,i_n^+})^{-1}.$$

We then apply Proposition 9.2 with $c = 0$, $\rho_1 = 0$, $\rho_2 = q_{n+1}^{-6}$, $\delta = q_{n+1}^{-7}$, $\nu = \exp(-q_{n+1}^{1-\beta})$ (*cf.* (11.363), (11.355)), $\varepsilon = q_{n+1}^{-\min(m/3,m-27)}$ (*cf.* (11.361), (11.354)) (estimates (9.282) and (9.283) are satisfied since $\overline{C} q_{n+1}^{-\min(m/3,m-27)} \leq q_{n+1}^{-7}/4 \ll q_{n+1}^{-6}$ and $q_{n+1}^{7\,\overline{a}_4} \exp(-q_{n+1}^{1/2}) \ll 1$).

Estimate (11.368) is a consequence of Proposition 9.2 applied to (11.353) and (11.356)

$$g_{i_n^-,i_n^+} \circ \Phi_{\Omega_{i_n^+}} \circ f_{F_{i_n^+}} \circ (g_{i_n^-,i_n^+})^{-1} = g_n^{\mathrm{HJ}} \circ \Phi_{\Omega_n^{\mathrm{HJ}}} \circ f_{F_n^{\mathrm{HJ}}} \circ (g_n^{\mathrm{HJ}})^{-1}$$

with $\mathbf{A}(c;\rho_1,\rho_2) = \widehat{D}_n \setminus (1/10)\widehat{D}_n$, $c = c_n$, $\rho_1 = |c_n|/40$, $\rho_2 = |c_n|/4$, $\delta_n = |c_n|/10$, $\nu = \exp(-q_{n+1}^{1/5})$ (*cf.* (11.359), (11.355)), $\varepsilon = \rho_n^{m/9}$ (*cf.* (11.354), (11.357)). Estimates (9.282) and (9.283) are satisfied since $\overline{C}\rho_n^{m/9} \leq |c_n|/40 \ll |c_n|/5$ (*cf.* (11.350)) and $\overline{C}(|c_n|/10)^{-\overline{a}_4} \exp(-q_{n+1}^{(1/4)-}) < 1$ (recall that $|c_n| \asymp (q_n q_{n+1})^{-1}$). □

12. Estimates on the measure of the set of KAM circles

We refer to Section 4.4 for the notations of this section. We observe

$$(W_{h,U})_{\mathbf{R}} := W_{h,U} \cap M_{\mathbf{R}} = W_{U\cap\mathbf{R}} := \{r \in U \cap \mathbf{R}\} \cap M_{\mathbf{R}}.$$

In particular in the (AA)-case $(W_{h,U})_{\mathbf{R}} = W_{U\cap\mathbf{R}} = \mathbf{T} \times (U \cap \mathbf{R})$ and in the (CC*)-case $(W_{h,U})_{\mathbf{R}} = W_{U\cap\mathbf{R}} = \{(x,y) \in \mathbf{R}^2,\ (1/2)(x^2+y^2) \in U \cap \mathbf{R}_+\}$.

12.1. *Classical KAM estimates.* — We first state a variant of the classical KAM theorem on abundance of invariant circles which is a consequence of Propositions 7.1, 7.2, 7.5 and Remark 7.1 on KAM Normal Forms.

In the next theorem the constant $\overline{a}_0$ is the one of Proposition 5.5 and the constant $\overline{a}_2$ was defined in Section 7 by (7.160).

Theorem **12.1.** — *Let* U *be a holed domain with disjoint holes* $D \in \mathcal{D}(U)$ *such that*

$$\sum_{D\in\mathcal{D}(U)} |D \cap \mathbf{R}|^{1/2} \leq 1 \qquad (\mathbf{12.369})$$

and $\Omega \in \widetilde{\mathcal{O}}_\sigma(U) \cap \mathcal{TC}(A,B)$ *(cf. (7.158)) with* A, B *satisfying (2.60),* $F \in \mathcal{O}_\sigma(W_{h,U})$

$$\overline{\varepsilon} := \|F\|_{W_{h,U}} \leq \underline{d}(U)^{\overline{a}_2}.$$

Then, if $f = \Phi_\Omega \circ f_F$ *one has*

$$\mathrm{Leb}_{M_{\mathbf{R}}}(W_{e^{-1/10}U\cap\mathbf{R}} \setminus \mathcal{L}(f, W_{U\cap\mathbf{R}})) \lesssim (\|F\|_{W_{h,U}})^{1/(2(\overline{a}_0+3))}.$$

Proof. — See Appendix J.2. □

Notation **12.2.** — *We define for* $\rho > 0$, $\mathbf{D}_{\mathbf{R}}(0,\rho) = \mathbf{D}(0,\rho) \cap \mathbf{R} =]-\rho, \rho[$ *and*

$$\widetilde{m}_f(\rho) = \mathrm{Leb}_{M_{\mathbf{R}}}(W_{\mathbf{D}_{\mathbf{R}}(0,\rho)} \setminus \mathcal{L}(f, \mathbf{D}_{\mathbf{R}}(0, e^{1/2}\rho))).$$

12.2. *Estimates on the measure of the set of invariant circles:* ω_0 *Diophantine (AA) or (CC) Case.* — We use the notation of Section 10 and assume that (both in the (AA) or (CC)-cases) (10.294), (10.295) (10.296) hold. We denote

$$\mathcal{D}_\rho = \mathcal{D}(U_{i_+(\rho)}). \tag{12.370}$$

Theorem **12.3.** — *For any* $\beta > 0$, $\rho \ll_\beta 1$

$$\textit{(AA)-case} \qquad \widetilde{m}_{\Phi_\Omega \circ f_F}(\rho) \lesssim \exp(-(1/\rho)^{(1/\tau)-\beta}) + \sum_{D\in\mathcal{D}_\rho} |\check{D} \cap \mathbf{R}|.$$

$$\textit{(CC) or (CC*)-case} \qquad \widetilde{m}_{\Phi_\Omega \circ f_F}(\rho) \lesssim \exp(-(1/\rho)^{(1/\tau)-\beta}) + \sum_{D\in\mathcal{D}_\rho} |\check{D} \cap \mathbf{R}_+|.$$

Moreover, for any $D \in \mathcal{D}_\rho$ *one has*

$$|\check{D} \cap \mathbf{R}| \lesssim \exp(-(1/\rho)^{\frac{1}{1+\tau}-\beta}). \tag{12.371}$$

Proof. — If $S \subset \mathbf{C}$ we denote $S_{\mathbf{R}} = S \cap \mathbf{R}$ (if $c \in \mathbf{R}$, $\mathbf{D}_{\mathbf{R}}(c,t) = \mathbf{D}(c,t) \cap \mathbf{R} =]c-t, c+t[$).

Choose (*cf.* Lemma 10.5) $\rho' \in [e^{1/4}\rho, e^{1/3}\rho]$ ($\rho \ll 1$) such that

$$U^{(\rho')} := \mathbf{D}(0,\rho') \cap U_{i_+(\rho)} = \mathbf{D}(0,\rho') \setminus \bigcup_{\substack{D\in\mathcal{D}(U_{i_+(\rho)})\\ D\subset\mathbf{D}(0,\rho')}} D$$

hence

$$e^{-1/10}\mathbf{D}_{\mathbf{R}}(0,\rho') \subset e^{-1/10}U^{(\rho')}_{\mathbf{R}} \cup \bigcup_{D\in\mathcal{D}_\rho} (1/4)\widehat{D}_{\mathbf{R}}. \tag{12.372}$$

Let us denote for short $f_i = \Phi_{\Omega_i} \circ f_{F_i}$, $f_D^{HJ} = \Phi_{\Omega_D^{HJ}} \circ f_{F_D^{HJ}}$ and

$$\mathcal{L}_{i_+(\rho)} = \mathcal{L}(f_{i_+(\rho)}, W_{U^{(\rho')}_{\mathbf{R}}}), \qquad \mathcal{L}_{\widehat{D}} = \mathcal{L}(f_D^{HJ}, W_{\widehat{D}_{\mathbf{R}} \setminus \check{D}_{\mathbf{R}}}).$$

We have from (10.318) (10.322) (10.316) of Proposition 10.7

$$(\mathbf{12.373})\qquad W_{h,U_{i_D}},\quad g^{-1}_{i_-(\rho),i_D}\circ f_{i_-(\rho)}\circ g_{i_-(\rho),i_D}=f_{i_D}$$

$$(\mathbf{12.374})\qquad W_{h/9,\widehat{D}\smallsetminus D},\quad (g^{HJ}_{\widehat{D}})^{-1}\circ f_{i_D}\circ g^{HJ}_{\widehat{D}}=f^{HJ}_{D}$$

$$(\mathbf{12.375})\qquad W_{h,U_{i_+(\rho)}},\quad g^{-1}_{i_-(\rho),i_+(\rho)}\circ f_{i_-(\rho)}\circ g_{i_-(\rho),i_+(\rho)}=f_{i_+(\rho)}$$

$$(\mathbf{12.376})\qquad W_{h,U_{i_-(\rho)}},\quad g^{-1}_{1,i_-(\rho)}\circ f\circ g_{1,i_-(\rho)}=f_{i_-(\rho)}.$$

From Lemma 10.3, Remark 10.1 and estimate (10.321) on the one hand, and estimate (10.325) on the other hand, we see that we can apply Theorem 12.1 to $f_{i_+(\rho)}$ and f^{HJ}_{D} to get the following decompositions

$$(\mathbf{12.377})\qquad W_{e^{-1/10}U^{(\rho')}_{\mathbf{R}}}\smallsetminus\mathcal{L}_{i_+(\rho)}\subset B_{i_+(\rho)},\qquad W_{e^{-1/10}\widehat{D}_{\mathbf{R}}}\smallsetminus\mathcal{L}_{\widehat{D}}\subset B_{\widehat{D}\smallsetminus\breve{D}}\cup E_{\breve{D}}$$

with $B_{\widehat{D}\smallsetminus\breve{D}}\subset W_{\widehat{D}_{\mathbf{R}}\smallsetminus\breve{D}_{\mathbf{R}}}$, $E_{\breve{D}}=W_{e^{-1/10}\breve{D}_{\mathbf{R}}}$ and

$$(\mathbf{12.378})\qquad \max\Big(\mathrm{Leb}_{M_{\mathbf{R}}}(B_{i_+(\rho)}),\,\mathrm{Leb}_{M_{\mathbf{R}}}(B_{\widehat{D}\smallsetminus\breve{D}})\Big)\lesssim\exp(-(1/\rho)^{(1/\tau)-\beta/2})$$

$$(\mathbf{12.379})\qquad \mathrm{Leb}_{M_{\mathbf{R}}}(E_{\breve{D}})\lesssim\mathrm{Leb}_{M_{\mathbf{R}}}(W_{e^{-1/10}\breve{D}_{\mathbf{R}}}).$$

We now introduce

$$(\mathbf{12.380})\qquad \widetilde{\mathcal{L}}_{i_+(\rho)}:=g_{i_-(\rho),i_+(\rho)}(\mathcal{L}_{i_+(\rho)}),\qquad \widetilde{\mathcal{L}}_{\widehat{D}}:=g_{i_-(\rho),i_D}\circ g^{HJ}_{D}(\mathcal{L}_{\widehat{D}})$$

$$(\mathbf{12.381})\qquad \widetilde{B}_{i_+(\rho)}=g_{i_-(\rho),i_+(\rho)}(B_{i_+(\rho)}),\quad \widetilde{B}_{\widehat{D}\smallsetminus\breve{D}}=g_{i_-(\rho),i_D}\circ g^{HJ}_{D}(B_{\widehat{D}\smallsetminus\breve{D}}),$$

$$(\mathbf{12.382})\qquad \widetilde{E}_{\breve{D}}=g_{i_-(\rho),i_D}\circ g^{HJ}_{D}(E_{\breve{D}}).$$

Lemma **12.4.** — *One has*

$$g_{i_-(\rho),i_+(\rho)}(W_{e^{-1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')})\smallsetminus\widetilde{\mathcal{L}}\subset\widetilde{B}$$

with

$$\widetilde{\mathcal{L}}=\widetilde{\mathcal{L}}_{i_+(\rho)}\cup\bigcup_{D\in\mathcal{D}_\rho}\widetilde{\mathcal{L}}_{\widehat{D}}\qquad \widetilde{B}=\widetilde{B}_{i_+(\rho)}\cup\bigcup_{D\in\mathcal{D}_\rho}(\widetilde{B}_{\widehat{D}\smallsetminus\breve{D}}\cup\widetilde{E}_{\breve{D}}).$$

Proof. — We observe that from (12.372) one has

$$(\mathbf{12.383})\qquad W_{e^{-1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')}\subset W_{e^{-1/10}U^{(\rho')}_{\mathbf{R}}}\cup\bigcup_{D\in\mathcal{D}_\rho}W_{(1/4)\widehat{D}_{\mathbf{R}}}$$

hence

$$g_{i_-(\rho),i_+(\rho)}(W_{e^{-1/10}\mathbf{D_R}(0,\rho')})$$
$$\subset g_{i_-(\rho),i_+(\rho)}(W_{e^{-1/10}U_{\mathbf{R}}^{(\rho')}}) \cup \bigcup_{D\in\mathcal{D}_\rho} g_{i_-(\rho),i_+(\rho)}(W_{(1/4)\widehat{D}_{\mathbf{R}}}).$$

Note that by Proposition 10.7 one has $\max(\|g_{i_-(\rho),i_+(\rho)} - id\|_{C^1}, \|g^{HJ}_{\widehat{D}} - id\|_{C^1}) \le \overline{\varepsilon}^{1/9}_{i_-(\rho)} \ll \widehat{K}^{-1}_{i_D}$ (since $N_{i_D} \le N^2_{i_-(\rho)}$) hence

$$g_{i_-(\rho),i_+(\rho)}(W_{(1/4)\widehat{D}_{\mathbf{R}}}) \subset W_{(1/2)\widehat{D}_{\mathbf{R}}} \subset g_{i_-(\rho),i_D} \circ g^{HJ}_{\widehat{D}}(W_{e^{-1/10}\widehat{D}_{\mathbf{R}}})$$

which yields

$$g_{i_-(\rho),i_+(\rho)}(W_{e^{-1/10}\mathbf{D_R}(0,\rho')})$$
$$\subset g_{i_-(\rho),i_+(\rho)}(W_{e^{-1/10}U_{\mathbf{R}}^{(\rho')}}) \cup \bigcup_{D\in\mathcal{D}_\rho} \left(g_{i_-(\rho),i_D} \circ g^{HJ}_{\widehat{D}}(W_{e^{-1/10}\widehat{D}_{\mathbf{R}}\smallsetminus\check{D}_{\mathbf{R}}}) \cup \widetilde{E}_{\check{D}}\right).$$

We then conclude using (12.377) and the notations (12.380). □

Lemma **12.5.** — *For some* $G \subset W_{e^{1/10}\mathbf{D_R}(0,\rho')}$ *one has* $\widetilde{\mathcal{L}} = \mathcal{L}(f_{i_-(\rho)}, G)$ *and*

$$\text{Leb}_{M_{\mathbf{R}}}(\widetilde{B}) \lesssim \exp(-(1/\rho)^{(1/\tau)-\beta}) + \sum_{D\in\mathcal{D}_\rho} \text{Leb}_{M_{\mathbf{R}}}(W_{e^{-1/10}\check{D}_{\mathbf{R}}}). \tag{12.384}$$

Proof. — We observe that from (4.86)

$$\widetilde{\mathcal{L}}_{i_+(\rho)} := g_{i_-(\rho),i_+(\rho)}(\mathcal{L}_{i_+(\rho)}) = \mathcal{L}(f_{i_-(\rho)}, g_{i_-(\rho),i_+(\rho)}(W_{U_{\mathbf{R}}^{(\rho')}})) \tag{12.385}$$

$$\widetilde{\mathcal{L}}_{\widehat{D}} := g_{i_-(\rho),i_D} \circ g^{HJ}_{D}(\mathcal{L}_{\widehat{D}}) = \mathcal{L}(f_{i_-(\rho)}, g_{i_-(\rho),i_D} \circ g^{HJ}_{D}(W_{\widehat{D}_{\mathbf{R}}\smallsetminus\check{D}_{\mathbf{R}}})) \tag{12.386}$$

hence,

$$\widetilde{\mathcal{L}} = \mathcal{L}(f_{i_-(\rho)}, G)$$

with

$$G = g_{i_-(\rho),i_+(\rho)}(W_{U_{\mathbf{R}}^{(\rho')}}) \cup \bigcup_{D\in\mathcal{D}_\rho} g_{i_-(\rho),i_D} \circ g^{HJ}_{D}(W_{\widehat{D}_{\mathbf{R}}\smallsetminus\check{D}_{\mathbf{R}}})$$

and clearly $G \subset W_{e^{1/10}\mathbf{D_R}(0,\rho')}$.

To get the estimate on the measure of $\widetilde{B}$ we use (12.378) and (12.381) to get

$$\text{Leb}_{M_{\mathbf{R}}}(\widetilde{B}_{i_+(\rho)}) \lesssim \exp(-(1/\rho)^{(1/\tau)-\beta/2}),$$

and (remember (10.309), (10.304), (10.305))

$$\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}\Big(\bigcup_{\mathrm{D}\in\mathcal{D}_\rho}\widetilde{\mathrm{B}}_{\widehat{\mathrm{D}}\smallsetminus\breve{\mathrm{D}}}\Big)\lesssim \mathrm{N}^2_{i_+(\rho)}\exp(-(1/\rho)^{(1/\tau)-\beta/2}) \lesssim \exp(-(1/\rho)^{(1/\tau)-\beta});$$

moreover (see (12.379), (12.382))

$$\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}\Big(\bigcup_{\mathrm{D}\in\mathcal{D}_\rho}\widetilde{\mathrm{E}}_{\breve{\mathrm{D}}}\Big)\lesssim \sum_{\mathrm{D}\in\mathcal{D}_\rho}\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}(\mathrm{W}_{e^{-1/10}\breve{\mathrm{D}}_{\mathbf{R}}}).$$

Summing up these estimates yields the desired inequality on the measure of $\widetilde{\mathrm{B}}$. □

End of the proof of Theorem 12.3.

Lemmata 12.4 and 12.5 give

$$g_{i_-(\rho),i_+(\rho)}(\mathrm{W}_{\mathbf{D}_{\mathbf{R}}(0,\rho')})\smallsetminus\mathcal{L}(f_{i_-(\rho)},\mathrm{W}_{e^{1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')})\subset\widetilde{\mathrm{B}}$$

hence

$$g_{1,i_-(\rho)}\circ g_{i_-(\rho),i_+(\rho)}(\mathrm{W}_{\mathbf{D}_{\mathbf{R}}(0,\rho')})\smallsetminus g_{1,i_-(\rho)}(\mathcal{L}(f_{i_-(\rho)},\mathrm{W}_{e^{1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')}))\subset g_{1,i_-(\rho)}(\widetilde{\mathrm{B}}). \tag{12.387}$$

Since the conjugation relation $g^{-1}_{1,i_-(\rho)}\circ f\circ g_{1,i_-(\rho)}=f_{i_-(\rho)}$ holds on $\mathrm{W}_{\mathrm{U}_{i_-(\rho)}\cap\mathbf{R}}$ (*cf.* (12.376)) and since $\mathrm{W}_{e^{1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')}\subset\mathrm{W}_{\mathrm{U}_{i_-(\rho)}\cap\mathbf{R}}$ (recall that by definition (10.300) $\mathbf{D}(0,2\rho)\subset\mathrm{W}_{h,\mathrm{U}_{i_-(\rho)}}$ and that $\|g_{1,i_-(\rho)}-id\|\ll\rho$) one has by (4.86)

$$\mathcal{L}(f,g_{1,i_-(\rho)}(\mathrm{W}_{e^{1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')}))=g_{1,i_-(\rho)}\mathcal{L}(f_{i_-(\rho)},\mathrm{W}_{e^{1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')}).$$

Equation (12.387) then implies that

$$g_{1,i_-(\rho)}\circ g_{i_-(\rho),i_+(\rho)}(\mathrm{W}_{\mathbf{D}_{\mathbf{R}}(0,\rho')})\smallsetminus\mathcal{L}(f,g_{1,i_-(\rho)}(\mathrm{W}_{e^{1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')}))\subset g_{1,i_-(\rho)}(\widetilde{\mathrm{B}}).$$

Finally, inclusions $\mathrm{W}_{\mathbf{D}_{\mathbf{R}}(0,\rho)}\subset g_{1,i_-(\rho)}\circ g_{i_-(\rho),i_+(\rho)}(\mathrm{W}_{\mathbf{D}_{\mathbf{R}}(0,\rho')})$ and $g_{1,i_-(\rho)}(\mathrm{W}_{e^{1/10}\mathbf{D}_{\mathbf{R}}(0,\rho')})\subset\mathrm{W}_{e^{1/2}\mathbf{D}(0,\rho)}$ yield

$$\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}(\mathrm{W}_{\mathbf{D}_{\mathbf{R}}(0,\rho)}\smallsetminus\mathcal{L}(f,\mathrm{W}_{e^{1/2}\mathbf{D}(0,\rho)}))\lesssim\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}(\widetilde{\mathrm{B}}).$$

We conclude by using the estimate (12.384) and the fact that

$$\mathrm{Leb}_{\mathrm{M}_{\mathbf{R}}}(\mathrm{W}_{e^{-1/10}\breve{\mathrm{D}}_{\mathbf{R}}})\le\begin{cases}|\breve{\mathrm{D}}\cap\mathbf{R}|, & \text{(AA)-case}\\ |\breve{\mathrm{D}}\cap\mathbf{R}_+|, & \text{(CC) or (CC*)-case.}\end{cases}$$

Proof of estimate (12.371) on the size of the holes.

Referring to (12.374) and (8.206) of Proposition 8.1 we see that

$$|\check{\mathrm{D}}| \lesssim \|f_{\mathrm{F}_{i_\mathrm{D}}}\|^{1/33}_{\mathrm{W}_{h,\mathrm{U}_{i_\mathrm{D}}}}$$

where $i_-(\rho) \le i_\mathrm{D} \le i_+(\rho)$. From (7.163)

$$\begin{aligned}\|f_{\mathrm{F}_{i_\mathrm{D}}}\|^{1/33}_{\mathrm{W}_{h,\mathrm{U}_{i_\mathrm{D}}}} &\le \overline{\varepsilon}^{1/33}_{i_\mathrm{D}} = e^{-h\mathrm{N}_{i_\mathrm{D}}/(33(\ln \mathrm{N}_{i_\mathrm{D}})^2)}\\ &\lesssim e^{-\mathrm{N}^{1-}_{i_\mathrm{D}}}\\ &\lesssim e^{-\mathrm{N}^{1-}_{i_-(\rho)}} \qquad (\text{since } i_-(\rho)\le i_\mathrm{D})\end{aligned}$$

hence from (10.302) for any $\beta > 0$

$$|\check{\mathrm{D}}| \lesssim \exp(-(1/\rho)^{(1/\iota(\rho))(1-\beta)}).$$

Using (10.303) we then get if $\rho \ll_\beta 1$ ($\check{\mathrm{D}}$ is a disk centered on the real axis)

$$|\check{\mathrm{D}} \cap \mathbf{R}| \lesssim \exp(-(1/\rho)^{\frac{1}{1+\tau}-\beta}).$$ □

12.3. *Estimates on the measure of the set of invariant circles: ω_0 Liouvillian, (CC)-Case.* — We now assume that (11.335), (11.336) (11.337) hold.

Theorem **12.6.** — *Let $\rho_n = (10\mathrm{A})/(q_n q_{n+1})$ and assume that $q_{n+1} \ge q_n^{10}$. Then, for all $\beta \ll 1$ and $n \gg_\beta 1$ one has*

$$\widetilde{m}_{\Phi_\Omega \circ f_\mathrm{F}}(\rho_n) \lesssim \exp(-q_{n+1}^{1/4-\beta}) + |(\check{\mathrm{D}}_n \cap \mathbf{R}_+)|.$$

Moreover,

$$|(\check{\mathrm{D}}_n \cap \mathbf{R}_+)| \lesssim e^{-q_n^{1-\beta}}. \tag{12.388}$$

Proof. — The principle of the proof is the same as that of Proposition 12.3 with the following modifications in the notations: we set $f_{i_n^\pm} = f_{\mathrm{F}_{i_n^\pm}}, f^{\mathrm{HJ}}_{\mathrm{D}_n} = f_{\mathrm{F}^{\mathrm{HJ}}_{\mathrm{D}_n}}$ and we replace in the proof the indices $i_\pm(\rho)$ by $i_n^\pm$, $\widehat{\mathrm{D}}$, D, $\check{\mathrm{D}}$ by $\widehat{\mathrm{D}}_n$, D_n $\check{\mathrm{D}}_n$, i_D by i_n^-, ρ by $(4/3)\rho_n$, ρ' by $2\rho_n$, $\mathrm{U}^{\rho'}$ by $\mathrm{U}^{(n)}$ and $\exp(-(1/\rho)^{(1/\tau)-\beta})$ by $\exp(-q_{n+1}^{(1/4)-})$. Instead of using the conjugation relations of Proposition 10.7 (Adapted Normal Forms in the (CC) or (CC*)-case) we use those of Proposition 11.3.

Estimate (12.388) is proved like (12.371) by noticing that

$$\|f_{\mathrm{F}_{i_\mathrm{D}}}\|^{1/33}_{\mathrm{W}_{h,\mathrm{U}_{i_{\mathrm{D}_n}}}} \lesssim \overline{\varepsilon}^{1/33}_{i_{\mathrm{D}_n}} \le \exp(-\mathrm{N}^{-(1-)}_{i_n-})$$

and using (11.347). □

Remark **12.1.** — Note that if the twist condition (11.335) is satisfied, then any twist condition $\mathcal{TC}(A', B)$ is satisfied with $A' \geq A$. We can thus replace in Theorem 12.6 $\rho_n = (10A)/(q_n q_{n+1})$ by $\rho_n = (10A')/(q_n q_{n+1})$ for any fixed $A' \geq A$ (then n has to be chosen larger).

13. Convergent BNF implies small holes

13.1. *Case where ω_0 Diophantine in the (AA) of (CC) setting.* — We keep here the notations of Sections 10 and 12.2, in particular we assume ω_0 is τ-Diophantine and that (10.294), (10.295), (10.296) hold.

Lemma **13.1.** — *If* $\mathrm{BNF}(\Phi_\Omega \circ f_F)$ *converges and is equal to a holomorphic function* $\Xi \in \mathcal{O}(\mathbf{D}(0,1))$ *then for all* $\beta > 0$, $\rho \ll_\beta 1$ *and for any* $D \in \mathcal{D}_\rho$

$$\|\Omega_{i_+(\rho)} - \Xi\|_{\widehat{D} \smallsetminus (1/10)\widehat{D}} \lesssim \exp\Big(-(1/\rho)^{(1/\tau)-\beta}\Big). \tag{13.389}$$

As a corollary, for any $D \in \mathcal{D}_\rho$ *and* $\gamma_D \leq \widehat{K}_{i_D}^{-2}$

$$\|\Omega^{\mathrm{HJ}}_{\widehat{D}} - \Xi(\cdot - \gamma_{\widehat{D}})\|_{(4/5)\widehat{D} \smallsetminus (1/5)\widehat{D}} \lesssim \exp\Big(-(1/\rho)^{(1/\tau)-\beta}\Big). \tag{13.390}$$

Proof. — Let us prove inequality (13.389). From (10.333) and Proposition 6.7 one gets

$$\begin{aligned}\|\Omega_{i_+(\rho)} - \Xi\|_{(1/2)\mathbf{D}(0,\rho^{b_\tau})} &\lesssim \exp(-(1/\rho)^{(1/\tau)-\beta/2}) + \exp(-(1/\rho)^{1-\beta}) \\ &\lesssim \exp(-(1/\rho)^{(1/\tau)-\beta/2}).\end{aligned}$$

Since the function $\Omega_{i_+(\rho)} - \Xi$ is holomorphic on $U^{(\rho)}$ and since the triple $(U_{i_+(\rho)}, \widehat{D} \smallsetminus (1/10)\widehat{D}, \mathbf{D}(0, \rho^{b_\tau}/2))$ is $(10 b_\tau)^{-1}|\ln\rho|^{-1}$-good, *cf.* Proposition 10.4, we have by Definition 3.3

$$\begin{aligned}\|\Omega_{i_+(\rho)} - \Xi\|_{\widehat{D} \smallsetminus (1/10)\widehat{D}} &\lesssim \exp\Big(-(10 b_\tau|\ln\rho|)^{-1}(1/\rho)^{(1/\tau)-\beta/2}\Big) \\ &\lesssim \exp(-(1/\rho)^{(1/\tau)-\beta}).\end{aligned}$$

The inequality (13.390) is then a consequence of (13.389) and (10.334). □

Corollary **13.2.** — *If* $\mathrm{BNF}(f) = \Xi$ *then for all* $\beta > 0$, $\rho \ll_\beta 1$, *and any* $D \in \mathcal{D}_\rho$ *the radius* $\rho_{\check{D}}$ *of the disk* $\check{D}$ *satisfies*

$$\rho_{\check{D}} \lesssim \exp\Big(-(1/\rho)^{(1/\tau)-\beta}\Big).$$

Proof. — This results from (13.390) and the Extension Property in Proposition 8.1. □

Corollary **13.3.** — *If* $\mathrm{BNF}(\Phi_\Omega \circ f_{\mathrm{F}})$ *converges then for all* $\beta > 0$, $\rho \ll_\beta 1$ *(recall Notation 12.2 for* $\widetilde{m}$*)*

$$\widetilde{m}_{\Phi_\Omega \circ f_{\mathrm{F}}}(\rho) \lesssim \exp\Big(-(1/\rho)^{(1/\tau)-\beta}\Big).$$

Proof. — This is a consequence of the previous Corollary 13.2 and of Proposition 12.3 since $\#\mathcal{D}_\rho \lesssim \rho^{1-2(\mu/\iota(\rho))} \lesssim \rho^{-1}$ (*cf.* (10.309), (10.304), (10.305)). □

13.2. *Case* ω_0 *is irrational in the (CC) setting.* — The notations here are those of Section 11. In particular we assume that (11.335), (11.336), (11.337) hold.

Lemma **13.4.** — *If* $\mathrm{BNF}(\Phi_\Omega \circ f_{\mathrm{F}})$ *converges and is equal to* $\Xi \in \mathcal{O}(\mathbf{D}(0,1))$ *then for all* $\beta \ll 1$, $n \gg_\beta 1$ *such that* $q_{n+1} \geq q_n^{10}$

$$\|\Omega_{i_n^+} - \Xi\|_{\widehat{\mathrm{D}}_n \setminus (1/10)\widehat{\mathrm{D}}_n} \lesssim \exp(-q_{n+1}^{1-\beta}). \tag{13.391}$$

As a corollary, for $\gamma_n \lesssim q_{n+1}^{-m} \lesssim |c_n|^{m/2}$

$$\|\Omega^{\mathrm{HJ}}_{\widehat{\mathrm{D}}_n} - \Xi(\cdot - \gamma_n))\|_{(4/5)\widehat{\mathrm{D}}_n \setminus (1/5)\widehat{\mathrm{D}}_n} \lesssim \exp(-q_{n+1}^{(1/4)-\beta}). \tag{13.392}$$

Proof. — Let us prove (13.391). From (11.367) and Proposition 6.7 one gets

$$\|\Omega_{i_n^+} - \Xi\|_{\mathbf{D}(0, q_{n+1}^{-6}/2)} \lesssim \exp(-q_{n+1}^{1-\beta/2}).$$

Since the function $\Omega_{i_n^+} - \Xi$ is holomorphic on $\mathrm{U}^{(n)}$ and since the triple $(\mathrm{U}^{(n)}, \widehat{\mathrm{D}}_n \setminus (1/10)\widehat{\mathrm{D}}_n, \mathbf{D}(0, q_{n+1}^{-6}/2))$ is $1/(10|\ln \rho_n|)$-good (see Proposition 11.2, Item (5)), we have by Definition 3.3 (remember (11.341))

$$\begin{aligned}\|\Omega_{i_n^+} - \Xi\|_{\widehat{\mathrm{D}}_n \setminus (1/10)\widehat{\mathrm{D}}_n} &\lesssim \exp\Big(-(10|\ln \rho_n|)^{-1}(q_{n+1})^{1-\beta/2}\Big)\\ &\lesssim \exp(-q_{n+1}^{1-\beta}).\end{aligned}$$

The inequality (13.392) is then a consequence of (13.391) and (11.368). □

Corollary **13.5.** — *If* $\mathrm{BNF}(f) = \Xi$ *then for any* $\beta > 0$, $n \gg_\beta 1$ *such that* $q_{n+1} \geq q_n^{10}$, *the radius* $\rho_{\check{\mathrm{D}}_n}$ *of the disk* $\check{\mathrm{D}}_n$ *satisfies*

$$\rho_{\check{\mathrm{D}}_n} \lesssim \exp(-q_{n+1}^{(1/4)-\beta}).$$

Proof. — This results from (13.392) and the Extension Property of Proposition 8.1. □

Corollary **13.6.** — *If* $\mathrm{BNF}(\Phi_\Omega \circ f_{\mathrm{F}})$ *converges, then for any* $\beta > 0$, $\mathrm{A}' \geq \mathrm{A}$ *and* $n \gg_{\beta,\mathrm{A}'} 1$ *such that* $q_{n+1} \geq q_n^{10}$ *one has*

$$\widetilde{m}_{\Phi_\Omega \circ f_{\mathrm{F}}}(\rho_n) \lesssim \exp(-q_{n+1}^{(1/4)-\beta}), \qquad \rho_n = 10\mathrm{A}'/(q_{n+1}q_n).$$

Proof. — This follows from the previous Corollary 13.5 and Proposition 12.6 and Remark 12.1. □

14. Proof of Theorems C, A and A'

14.1. *Proof of Theorem C.*

14.1.1. *(AA) Case.* — Let $f(\theta, r) = (\theta + \omega_0, r) + (\mathrm{O}(r), \mathrm{O}^2(r))$ be a real analytic symplectic diffeomorphism of the annulus $\mathbf{T} \times [-1, 1]$ satisfying the twist condition (1.14). We can perform some steps of the classical Birkhoff Normal Form procedure, Proposition 6.2: for some $h > 0$, $\rho_0 > 0$, there exists $\widetilde{g} = f_{\widetilde{\mathrm{Z}}} = id + (\mathrm{O}(r), \mathrm{O}(r^2))$, $\widetilde{\Omega} \in \mathcal{O}_\sigma(e^{10h}\mathbf{D}(0, \rho_0))$, $\mathrm{Z}, \mathrm{F} \in \mathcal{O}_\sigma(e^{10h}(\mathbf{T}_h \times \mathbf{D}(0, \rho_0))) \cap \mathrm{O}(r^2)$, such that on $e^{10h}(\mathbf{T}_h \times \mathbf{D}(0, \rho_0))$ one has

$$\widetilde{g}^{-1} \circ f \circ \widetilde{g} = \Phi_{\widetilde{\Omega}} \circ f_{\mathrm{F}},$$

$$\forall 0 \leq \rho \leq \rho_0, \quad \|\mathrm{F}\|_{e^{10h}(\mathbf{T}_h \times \mathbf{D}(0,\rho))} \leq \rho^m$$

$$(2\pi)^{-1}\widetilde{\Omega}(r) = \omega_0 r + b_2(f) r^2 + \mathrm{O}(r^3)$$

$$\widetilde{\mathrm{Z}}(\theta, r) = \sum_{j=2}^{9} \widetilde{\mathrm{Z}}_j(\theta) r^j + r^{10}\widetilde{\mathrm{Z}}_{\geq 10}(\theta, r)$$

where m is the constant appearing in (10.296). Applying Lemma 2.5 to $\widetilde{\Omega}(r)$ and Lemma 2.2 to $r^{10}\widetilde{\mathrm{Z}}_{\geq 10}(\theta, r)$ we can find, for some $0 < \overline{\rho} \ll \rho_0$, C^3 Whitney extensions $\Omega \in \widetilde{\mathcal{O}}_\sigma(e^{10h}\mathbf{D}(0, \overline{\rho}))$ and $\mathrm{Z} \in \widetilde{\mathcal{O}}_\sigma(e^{10h}\mathrm{W}_{h,\mathbf{D}(0,\overline{\rho})})$ of $(\widetilde{\Omega}, e^{10h}\mathbf{D}(0, \overline{\rho}))$ and $(\widetilde{\mathrm{Z}}, e^{h/10}\mathrm{W}_{h,\mathbf{D}(0,\overline{\rho})})$ such that $g := f_{\mathrm{Z}} \in \widetilde{\mathrm{Symp}}_{ex,\sigma}(e^{h/10}\mathrm{W}_{h,\mathbf{D}(0,\overline{\rho})})$ (see Notations 2.3, 2.6 and 4.8),

$$\Omega \in \mathcal{TC}(\mathrm{A}, \mathrm{B}), \quad \mathrm{A} = 3\min(b_2(f), b_2(f)^{-1}), \quad \mathrm{B} \geq 1 \tag{14.393}$$

$$g(\{r = 0\}) = (\{r = 0\}), \qquad \|g - id\|_{\mathrm{C}^1} \leq 1/100. \tag{14.394}$$

Since

$$g^{-1} \circ f \circ g = \Phi_\Omega \circ f_{\mathrm{F}} \qquad [e^{h/10}\mathrm{W}_{h,\mathbf{D}(0,\overline{\rho})}]$$

one has from (4.86), for any $\rho \le \overline{\rho}$,

$$\mathcal{L}(f, g(W_{\mathbf{D}(0,\rho)})) = g(\mathcal{L}(\Phi_\Omega \circ f_F, W_{\mathbf{D}(0,\rho)}))$$

hence, using the fact that $g(\{r=0\}) = (\{r=0\})$ and $\|g - id\|_{C^1} \le 1/100$, we get the inequality

$$(\mathbf{14.395}) \qquad m_f(\rho) \lesssim \widetilde{m}_{\Phi_\Omega \circ f_F}(2\rho).$$

The first part of Theorem C is then a consequence of Theorem 12.3 applied to $\Phi_\Omega \circ f_F$ (which satisfies (10.294), (10.295) (10.296)): if we define $\breve{\mathcal{D}}_t$ as the set $\{\breve{D},\ D \in \mathcal{D}_{2t}\}$ (each $\breve{D}$ is associated to a $D \in \mathcal{D}_{2t}$), formula (1.23) comes from the fact that $\#\breve{\mathcal{D}}_t = \#\mathcal{D}_{2t} = \#\mathcal{D}(U_{i_+(2t)})$ (recall the notation (12.370)) and from (10.309), (10.304), (10.305)); on the other hand, (1.24) is a consequence of (12.371); finally (1.25) follows from Theorem 12.3 and inequality (14.395) (we take $\rho = t$).

The second part of Theorem C is a consequence of Corollary 13.2 because if the BNF of f converges, the same is true for that of $\Phi_\Omega \circ f_F$. □

14.1.2. *(CC) Case.* — Let f be a real analytic twist symplectic map of the real disk admitting the origin as an elliptic fixed point with Diophantine frequency ω_0, $(x,y) \mapsto \Phi_{2\pi\omega_0 r(x,y)}(x,y) + O^2(x,y)$, $r(x,y) = (1/2)(x^2+y^2)$ and satisfying the twist condition (1.14). We first make the symplectic change of variables (4.77) $(z,w) = \varphi(x,y)$,

$$\begin{cases} z = \frac{1}{\sqrt{2}}(x+iy) \\ w = \frac{i}{\sqrt{2}}(x-iy) \end{cases} \Longleftrightarrow \begin{cases} x = \frac{1}{\sqrt{2}}(z-iw) \\ y = \frac{-i}{\sqrt{2}}(z+iw) \end{cases}$$

and we write the thus obtained symplectic map $(z,w) \mapsto \widetilde{f}(z,w)$, $\widetilde{f} = \varphi \circ f \circ \varphi^{-1}$ as

$$\widetilde{f} = \Phi_{2\pi\omega_0 r} \circ f_{F_0}, \qquad r = -izw.$$

We observe that (*cf.* (4.86))

$$(\mathbf{14.396}) \qquad \mathcal{L}(f, W) = \mathcal{L}(\widetilde{f}, \varphi(W)).$$

Like in the (AA)-case (Section 14.1.1) we perform some steps of Birkhoff Normal Form, Proposition 6.1 and make some Whitney extensions (Lemma 2.2) to obtain for some $h>0$, $\overline{\rho} > 0$, maps $\Omega \in \widetilde{\mathcal{O}}_\sigma(e^{10h}\mathbf{D}(0,\overline{\rho}))$, $F \in \mathcal{O}_\sigma(e^{10h}W_{h,\mathbf{D}(0,\overline{\rho})})$, $g \in \widetilde{\mathrm{Symp}}_{ex,\sigma}(e^{h/10}W_{h,\mathbf{D}(0,\overline{\rho})})$ satisfying

$$(\mathbf{14.397}) \qquad g^{-1} \circ \widetilde{f} \circ g = \Phi_\Omega \circ f_F, \qquad [e^{10h}W_{h,\mathbf{D}(0,\overline{\rho})}]$$

$$(\mathbf{14.398}) \qquad g(\{r=0\}) = (\{r=0\}), \qquad \|g - id\|_{C^1} \le 1/100.$$

$$(\mathbf{14.399}) \qquad \Omega \in \mathcal{TC}(A,B), \quad A = 3\min(b_2(f), b_2(f)^{-1}), \quad B \ge 1$$

$$(\mathbf{14.400}) \qquad \forall\, \rho \leq \overline{\rho}, \quad \|F\|_{e^{10h}W_{h,\mathbf{D}(0,\rho)}} \leq \rho^m$$

where m is the constant appearing in (10.296).

Applying (14.396), (14.397), (14.398) yields for $\rho \leq \overline{\rho}$ (*cf.* (14.395))

$$(\mathbf{14.401}) \qquad m_f(\rho) \leq m_{\widetilde{f}}(2\rho) \lesssim \widetilde{m}_{\Phi_\Omega \circ f_F}(4\rho).$$

The conclusion of Theorem C is then obtained in the same way as in the previous Section 14.1.1. □

14.2. *Proof of Theorem A.* — The conclusion of Theorem A is an immediate consequence of (1.23), (1.25), (1.26) of Theorem C: for any $\beta > 0$ and $t \ll_\beta 1$

$$m_f(t) \lesssim \exp\Big(-(1/t)^{(1/\tau)-\beta}\Big). \qquad \square$$

14.3. *Proof of Theorem A'.* — We proceed like in the previous Section 14.1.2 to obtain (14.397)–(14.400) and then,

$$(\mathbf{14.402}) \qquad m_f(\rho) \leq m_{\widetilde{f}}(2\rho) \lesssim \widetilde{m}_{\Phi_\Omega \circ f_F}(4\rho).$$

We now apply Corollary 13.6 to $\Phi_\Omega \circ f_F$. Setting $\rho_n = 10A/(q_n q_{n+1})$ with $A = 3\min(b_2(f), b_2(f)^{-1})$ (*cf.* (14.399)) we get for any $\beta \ll 1$ and any $n \gg_\beta 1$ such that $q_{n+1} \geq q_n^{10}$, the inequality

$$\widetilde{m}_{\Phi_\Omega \circ f_F}(\rho_n) \lesssim \exp(-q_{n+1}^{(1/4)-\beta}).$$

Hence if $t_n := 5\min(b_2(f), b_2(f)^{-1})/(q_n q_{n+1}) \leq \rho_n/4$ one has (*cf.* (14.402))

$$m_f(t_n) \leq m_{\widetilde{f}}(2t_n) \lesssim \widetilde{m}_{\Phi_\Omega \circ f_F}(4t_n) \lesssim \exp(-q_{n+1}^{1/5}). \qquad \square$$

15. Creating hyperbolic periodic points

Let $\Omega \in \widetilde{\mathcal{O}}_\sigma(\mathbf{D}(0, \overline{\rho}))$ satisfy a twist condition ($A, B \geq 1$),

$$(\mathbf{15.403}) \qquad \forall\, r \in \mathbf{R}, \; A^{-1} \leq (1/2\pi)\partial^2\Omega(r) \leq A, \quad \text{and} \quad \|(1/2\pi)D^3\Omega\|_{\mathbf{C}} \leq B,$$

$\overline{a}_3 \in \mathbf{N}$, $\overline{a}_3 \geq 10$, be the constant appearing in Proposition G.1 of the Appendix and (p_n/q_n) the sequence of convergents of $\omega_0 = (2\pi)^{-1}\partial\Omega(0)$. We introduce for $n \geq 1$, the sequence c_n defined by

$$(\mathbf{15.404}) \qquad (2\pi)^{-1}\partial\Omega(c_n) = p_n/q_n, \qquad \frac{(2A)^{-1}}{q_n q_{n+1}} \leq |c_n| \leq \frac{A}{q_n q_{n+1}}.$$

Proposition **15.1.** — *Let $h > 0$, $n \in \mathbf{N}$ large enough and $F \in \mathcal{O}_\sigma(\mathbf{T}_h \times \mathbf{D}(c_n, |c_n|^2))$ such that*

$$e^{-q_n h} < |c_n|^{10}$$
$$\|F\|_{\mathbf{T}_h \times \mathbf{D}(c_n,|c_n|^2)} \lesssim |c_n|^{\overline{a}_3}$$

and

$$|\widehat{F}(q_n, c_n)| \geq e^{-q_n h} |c_n|^{\overline{a}_3 + 1/2}.$$

Then,

$$m_{\Phi_\Omega \circ f_F}(c_n) \geq C_h^{-1} |c_n|^{2\overline{a}_3+1} e^{-4q_n h}.$$

The constant C_h can be chosen to be non increasing w.r.t. h.

This proposition will be a consequence of the more precise statement given by the following Proposition 15.2.

For $p \in \mathbf{Z}$, $q \in \mathbf{N}^*$, $p \wedge q = 1$, p/q small enough, there exists a unique $c_{p/q} \in \mathbf{D}(0, \overline{\rho}) \cap \mathbf{R}$ such that

$$\omega(c_{p/q}) := (2\pi)^{-1} \partial\Omega(c_{p/q}) = p/q.$$

We define

$$\rho_{p/q} = \min(|c_{p/q}/4|, q^{-9}) \tag{15.405}$$

and assume that

$$\varepsilon_{p/q} := \|F\|_{\mathbf{D}(c_{p/q},\rho_{p/q})} \leq |c_{p/q}|^{\overline{a}_3}. \tag{15.406}$$

The $\pm q$-th Fourier coefficients of $F(\cdot, r)$, $\widehat{F}(\pm q, r) = (2\pi)^{-1} \int_0^{2\pi} F(\theta, r) e^{\mp iq\theta} d\theta$ satisfy

$$|\widehat{F}(\pm q, r)| \lesssim e^{-qh} \varepsilon_{p/q}$$

and since F is σ-symmetric, for every $r \in \mathbf{D}(0, \overline{\rho}) \cap \mathbf{R}$, $\overline{\widehat{F}(q, r)} = \widehat{F}(-q, r)$.

Proposition **15.2.** — *Assume (15.406) is satisfied and*

$$e^{-qh} < \rho_{p/q}^{10} \tag{15.407}$$

$$|\widehat{F}(\pm q, c_{p/q})| = \nu_q e^{-qh} \varepsilon_{p/q}. \tag{15.408}$$

$$\nu_q^{-1} q \rho_{p/q} \leq 1/q. \tag{15.409}$$

Then, there exists in a neighborhood of $\mathbf{T} \times \{c_{p/q}\} \subset \mathbf{T} \times \mathbf{R}$ an open set of area $\geq C_h^{-1}(\nu_q \varepsilon_{p/q} e^{-qh})^{3/2}$, $C_h > 0$, that has an empty intersection with any possible (horizontal) invariant circle of the symplectic diffeomorphism $\Phi_\Omega \circ f_F$.

Remark **15.1.** — One can choose the constant C_h to be non increasing with respect to h.

Let us see how it provides a proof of Proposition 15.1.

Proof of Proposition 15.1. — Since for n large enough

$$e^{-q_n h} < \min(|c_n/4|, q_n^{-9})^{10}$$
$$|\widehat{F}(q_n, c_n)| \geq e^{-q_n h}|c_n|^{1/2}\|F\|_{\mathbf{T}_h\times\mathbf{D}(c_n,|c_n|^2)}$$
$$\lim_{n\to\infty} |c_n|^{-1/2} q_n \min(|c_n/4|, q_n^{-9}) = 0$$

we can apply Proposition 15.2 with $q = q_n$, $\varepsilon_{p/q} = \|F\|_{\mathbf{T}_h\times\mathbf{D}(c_n,|c_n|^2)}$, $\nu_q = |c_n|^{1/2}$, $c_{p/q} = c_n$, $\rho_{p/q} = \min(|c_n/4|, q_n^{-9})$. We then get

$$m_{\Phi_\Omega\circ f_F}(c_n) \gtrsim (|c_n|^{1/2} e^{-q_n h}\|F\|_{\mathbf{T}_h\times\mathbf{D}(c_n,|c_n|^2)})^2.$$

But, because $((2\pi)^{-1}\int_0^{2\pi}|F(\theta, c_n)|^2 d\theta)^{1/2} \geq |\widehat{F}(q_n, c_n)|$, one has $\|F\|_{\mathbf{T}_h\times\mathbf{D}(c_n,|c_n|^2)} \geq |\widehat{F}(q_n, c_n)|$, hence

$$m_{\Phi_\Omega\circ f_F}(c_n) \geq C_h^{-1}|c_n|^{2\overline{a}_3+1} e^{-4q_n h}. \qquad \square$$

The proof of Proposition 15.2 will occupy the next subsections.

15.1. *Putting the system into q-resonant Normal Form.* — Conditions (15.405) and (15.406) show that we can apply Proposition G.1: it provides us with the following q-resonant Normal Form

$$g_{\mathrm{RNF}}^{-1} \circ \Phi_\Omega \circ f_F \circ g_{\mathrm{RNF}} = \Phi_{2\pi(p/q)r} \circ \Phi_{\overline{\Omega}} \circ f_{\overline{F}^{res}} \circ f_{F^{cor}}$$

$$(\mathbf{15.410}) \qquad \begin{cases} \overline{\Omega} = \Omega - 2\pi(p/q)r + \mathcal{M}_0(F^{res}) \\ \overline{F}^{res} = F^{res} - \mathcal{M}_0(F^{res}) \end{cases}$$

$F^{res} \in \mathcal{O}_\sigma(\mathbf{T}_{h-1/q} \times \mathbf{D}(c_{p/q}, e^{-1/q}\rho_{p/q}))$, $\overline{F}^{res} = F^{res} - \mathcal{M}_0(F^{res})$; these last two functions are $1/q$-periodic (in the θ-variable) and are such that

$$\|F^{res}\|_{\mathbf{T}_{h-1/q}\times\mathbf{D}(c_{p/q},e^{-1/q}\rho_{p/q})} \lesssim \varepsilon_{p/q}$$
$$F^{res} = T_N^{res}(F + O(q\rho_{p/q}\|F\|_{W_{h,\mathbf{D}(c_{p/q},\rho_{p/q})}})).$$

Also,

$$(\mathbf{15.411}) \qquad \|F^{cor}\|_{e^{-1/q}W_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim \exp(-\overline{\rho}^{\,-1/4})\|F\|_{W_{h,\mathbf{D}(c,\overline{\rho})}},$$

$$(\mathbf{15.412}) \qquad \|g_{\mathrm{RNF}} - id\|_{C^1} \lesssim (q\overline{\rho}^{\,-2})^2\|F\|_{h,\mathbf{D}(c,\overline{\rho})} \leq \overline{\rho}^{\,\overline{a}_3-5}.$$

Lemma **15.3.** — *On* $\mathbf{T}_{1/q}\times \mathbf{D}(c_{p/q},e^{-1/q}\rho_{p/q}/2)$ *one has*

$$F^{res}(\theta,r)=u_0^{res}(r)+\sum_{\pm}u_{1,\pm}^{res}(r)e^{\pm iq\theta}+u_{\geq 2}^{res}(\theta,r) \tag{15.413}$$

where on $\mathbf{D}(c_{p/q},e^{-1/q}\rho_{p/q}/2)$ *one has*

$$\begin{aligned} u_0^{res}(r)&=\mathcal{M}_0(F^{res})=\widehat{F}(0,r)+O(q\rho_{p/q}\varepsilon_{p/q})\\ u_{1,\pm}^{res}(r)&=\widehat{F}(\pm q,r)+O(e^{-qh}q\rho_{p/q}\varepsilon_{p/q})=O(e^{-qh}\varepsilon_{p/q}) \end{aligned} \tag{15.414}$$

and

$$\|u_{\geq 2}^{res}\|_{\mathbf{T}_{1/q}\times\mathbf{D}(c_{p/q},e^{-1/q}\rho_{p/q}/2)}\lesssim e^{-2qh}\varepsilon_{p/q}.$$

Proof. — We recall that from (G.526)

$$F^{res}=T_N^{q-res}(F+G)$$

(see the notation (G.519) for T_N^{q-res}) where

$$\|G\|_{h-1/q,e^{-1/q}\rho_{p/q}/2}=O(q\rho_{p/q}\|F\|_{h,\mathbf{D}(c_{p/q},\rho_{p/q}/2)}).$$

Hence

$$|\widehat{G}(0,r)|\lesssim q\rho_{p/q}\varepsilon_{p/q},\qquad |\widehat{G}(\pm q,r)|\lesssim e^{-q(h-2/q)}q\rho_{p/q}\varepsilon_{p/q}\lesssim e^{-qh}q\rho_{p/q}\varepsilon_{p/q}. \tag{15.415}$$

On the other hand since $e^{-2q(h-3/q)}\lesssim e^{-2qh}$

$$\|T_N^{q-res}F-\widehat{F}(0,r)-\sum_{\pm}\widehat{F}(\pm q,r)e^{\pm iq\theta}\|_{1/q,\rho_{p/q}/2}\lesssim e^{-2qh}\varepsilon_{p/q}$$

and

$$\|T_N^{q-res}G-\widehat{G}(0,r)-\sum_{\pm}\widehat{G}(\pm q,r)e^{\pm iq\theta}\|_{1/q,e^{-1/q}\rho_{p/q}/2}\lesssim q\rho_{p/q}\varepsilon_{p/q}e^{-2qh}.$$

Summing these two inequalities and using (15.415) gives (15.413). □

With these notations

$$\begin{cases}\overline{\Omega}=\Omega-2\pi(p/q)r+u_0^{res}(r)\\ \overline{F}^{res}=F^{res}-u_0^{res}(r).\end{cases}$$

We denote by $\overline{c}\in\mathbf{R}$ the point where

$$\partial\overline{\Omega}(\overline{c})=0;$$

since $\|u_0^{res}\|_{\mathbf{D}(c_{p/q},\rho_{p/q})} \lesssim \varepsilon_{p/q}$ and Ω satisfies the twist condition (15.403) one has

$$\overline{c} = c + \mathrm{O}(\varepsilon_{p/q}) \in \mathbf{D}(c, (3/4)\rho_{p/q}),$$
$$\overline{\Omega}(r) = \mathrm{cst} + (\varpi/2)(r-c)^2 + \mathrm{O}((r-c)^3)$$

for some $\varpi \gtrsim \mathrm{A}^{-1}$. Since F^{res} is σ-symmetric we can write

$$\sum_{\pm} u_{1,\pm}^{res}(r)e^{\pm iq\theta} = a(r)\cos(q\theta) + b(r)\sin(q\theta)$$

and from (15.408), (15.414), (15.409) we can assume, shifting the variable $\theta \in \mathbf{T}_h$ by a translation $\theta \mapsto \theta + \alpha_c$ ($\alpha_c \in \mathbf{T}$) if necessary, that

(**15.416**) $$b(\overline{c}) = 0, \quad a(\overline{c}) = \overline{\nu}_q e^{-qh}\varepsilon_{p/q}, \quad \overline{\nu}_q = \nu_q - \mathrm{O}(q\rho_{p/q}) = \nu_q(1 + o_{q^{-1}}(1))$$

with

$$\max(\|a\|_{\mathbf{D}(c,\rho_{p/q})}, \|b\|_{\mathbf{D}(c,\rho_{p/q})}) \lesssim e^{-qh}\varepsilon_{p/q}.$$

Thus,

$$\begin{cases} \overline{\Omega}(r) = \Omega(r) - 2\pi(p/q)r + u_0^{res}(r) = \mathrm{cst} + (\varpi/2)(r-\overline{c})^2 + \mathrm{O}((r-\overline{c})^3) \\ \overline{\mathrm{F}}^{res}(\theta, r) = a(r)\cos(q\theta) + b(r)\sin(q\theta) + u_{\geq 2}^{res}(\theta, r). \end{cases}$$

15.2. *Coverings.* — Like in Section 8.2 (*cf.* (8.215)) we define

$$\widetilde{\Omega}^{res} \in \mathcal{O}_\sigma(\mathbf{D}(0, qe^{-2/q}\rho_{p/q}/2)), \qquad \widetilde{\mathrm{F}}^{res} \in \mathcal{O}_\sigma(\mathbf{T}_{qh-2} \times \mathbf{D}(0, qe^{-2/q}\rho_{p/q}/2))$$

(**15.417**) $$\begin{cases} \widetilde{\Omega}^{res}(r) = q^2\overline{\Omega}(\overline{c} + r/q) \\ \widetilde{\mathrm{F}}^{res}(\theta, r) = q^2\overline{\mathrm{F}}^{res}([\theta/q]_{\mathrm{mod}\ (2\pi/q)\mathbf{Z}}, \overline{c} + r/q) \end{cases}$$

hence

$$\begin{cases} \widetilde{\Omega}^{res}(r) = \mathrm{cst} + \varpi r^2/2 + \mathrm{O}(r^3) = \varpi r^2/2 + \widetilde{\omega}(r) \\ \widetilde{\mathrm{F}}^{res}(\theta, r) = \widetilde{a}(r)\cos(q\theta) + \widetilde{b}(r)\sin(q\theta) + \widetilde{u}_{\geq 2}^{res}(\theta, r) \end{cases}$$

with

$$\widetilde{a}(r) = q^2 a(\overline{c} + r/q), \qquad \widetilde{b}(r) = q^2 b(\overline{c} + r/q),$$
$$\widetilde{u}_{\geq 2}^{res}(\theta, r) = q^2 u_{\geq 2}^{res}(\theta/q, \overline{c} + r/q).$$

Let us define

(**15.418**) $$\begin{aligned} \widetilde{\mathrm{H}}^{res}(\theta, r) &:= \widetilde{\Omega}^{res}(r) + \widetilde{\mathrm{F}}^{res}(\theta, r) \\ &= \mathrm{cst} + (1/2)\varpi r^2 + \widetilde{a}(r)\cos\theta + \widetilde{b}(r)\sin\theta + \widetilde{\omega}(r) + \widetilde{u}_{\geq 2}^{res}(\theta, r). \end{aligned}$$

We make explicit the linear plus quadratic part $H_Q(\theta, r)$ of $(1/2)\varpi r^2 + \widetilde{a}(r)\cos\theta + \widetilde{b}(r)\sin\theta$ at $(\theta, r) = (0, 0) \in \mathbf{R}^2$ (recall that $\widetilde{b}(0) = 0$) which appears in

$$\begin{aligned}(1/2)\varpi r^2 + \widetilde{a}(r)\cos\theta + \widetilde{b}(r)\sin\theta \\ &= \widetilde{a}(0) + r\partial_r\widetilde{a}(0) - (\theta^2/2)\widetilde{a}(0) + (r^2/2)(\varpi + \partial_r^2\widetilde{a}(0)) \\ &\quad + r\theta\partial_r\widetilde{b}(0) + g_0(\theta, r)\end{aligned}$$

where $g_0(\theta, r) = O_3(\theta, r) = O(|\theta|^3 + |r|^3)$; we can then write

$$\widetilde{H}^{res}(\theta, r) = \text{cst} + H_Q(\theta, r) + \widetilde{\omega}(r) + g(\theta, r) \tag{15.419}$$

with

$$\begin{cases} H_Q(\theta, r) = \frac{1}{2}\left\langle Q\begin{pmatrix}\theta \\ r\end{pmatrix}, \begin{pmatrix}\theta \\ r\end{pmatrix}\right\rangle + \left\langle\begin{pmatrix}0 \\ \partial_r\widetilde{a}(0)\end{pmatrix}, \begin{pmatrix}\theta \\ r\end{pmatrix}\right\rangle \\ Q = \begin{pmatrix}-\widetilde{a}(0) & \partial_r\widetilde{b}(0) \\ \partial_r\widetilde{b}(0) & \varpi + \partial_r^2\widetilde{a}(0)\end{pmatrix}\end{cases} \tag{15.420}$$

and

$$g(\theta, r) = g_0(\theta, r) + \widetilde{u}^{res}_{\geq 2}(\theta, r), \qquad g_0(\theta, r) = O_3(\theta, r). \tag{15.421}$$

For further records we mention the following estimates

$$\max(\|\widetilde{a}\|_{\mathbf{D}(0, q\rho_{p/q})}, \|\widetilde{b}\|_{\mathbf{D}(0, q\rho_{p/q})}) \lesssim q^2 e^{-qh}\varepsilon_{p/q} \tag{15.422}$$

$$\widetilde{b}(0) = 0, \qquad \widetilde{a}(0) = q^2\overline{\nu}_q e^{-qh}\varepsilon_{p/q}, \qquad \text{where } \overline{\nu}_q = \nu_q(1 + o_{1/q}(1)) \tag{15.423}$$

$$\|\widetilde{u}^{res}_{\geq 2}\|_{\mathbf{T}_{1/q}\times\mathbf{D}(0, e^{-1/q}q\rho_{p/q}/2)} \lesssim q^2 e^{-2qh}\varepsilon_{p/q} \tag{15.424}$$

and for $(l_1, l_2) \in \mathbf{N}^2$, $l_1 + l_2 \leq 2$ and $0 < t < \rho_{p/q}/10$

$$\|\partial_\theta^{l_1}\partial_r^{l_2} g_0(\theta, r)\|_{\mathbf{D}(0,t)\times\mathbf{D}(0,t)} \lesssim q^2 t^{3-l_1-l_2} e^{-qh}\varepsilon_{p/q} \tag{15.425}$$

$$\|\partial_\theta^{l_1}\partial_r^{l_2} g(\theta, r)\|_{\mathbf{D}(0,t)\times\mathbf{D}(0,t)} \lesssim [q^2 t^{3-l_1-l_2} e^{-qh}\varepsilon_{p/q} + q^{l_1}\rho_{p/q}^{-l_2} q^2 e^{-2qh}\varepsilon_{p/q}]. \tag{15.426}$$

15.3. *Existence of a hyperbolic fixed point for $f_{H_Q+\widetilde{\omega}}$.* — We refer to Appendix N for the definition of the notion of a (κ, δ)-hyperbolic fixed point.

Lemma **15.4.** — *The affine symplectic map $f_{H_Q+\widetilde{\omega}(r)}$ has a (κ, δ)-hyperbolic fixed point $(\theta_0, r_0) \in \mathbf{D}(0, \rho_{p/q}^5)^2 \cap \mathbf{R}^2$ with*

$$\delta = \kappa = q(\varpi\nu_q\varepsilon_{p/q}e^{-qh})^{1/2}(1 + o_{1/q}(1))$$

with stable and unstable directions at this point of the form $\begin{pmatrix} 1 \\ m_\pm \end{pmatrix}$ *where*

$$m_\pm = \pm q(\nu_q e^{-qh}\varepsilon_{p/q}/\varpi)^{1/2}(1+o_{1/q}(1)).$$

Proof. — See Appendix N.2. □

15.4. *Stable and unstable manifolds of* $f_{\tilde{\mathrm{H}}^{res}}$.

Lemma **15.5.** — *The symplectic diffeomorphism* $f_{\tilde{\mathrm{H}}^{res}}$ *has a* (κ,δ)*-hyperbolic fixed point* $(\theta_1, r_1) \in \mathbf{D}(0,\rho_{p/q}^4)^2 \cap \mathbf{R}^2$ *with*

$$\kappa = \delta = q(\varpi\,\nu_q e^{-qh}\varepsilon_{p/q})^{1/2}(1+o_{1/q}(1)). \tag{15.427}$$

The stable and unstable directions at this point are of the form $\begin{pmatrix} 1 \\ m_\pm \end{pmatrix}$ *where*

$$m_\pm = \pm q(\nu_q e^{-qh}\varepsilon_{p/q}/\varpi)^{1/2}(1+o_{1/q}(1)).$$

Proof. — From (15.419),

$$\begin{aligned} f_{\tilde{\mathrm{H}}^{res}} &= f_{\mathrm{H_Q}+\tilde{\omega}+g} \\ &= f_{\mathrm{H_Q}+\tilde{\omega}} \circ f_{\tilde{g}}, \qquad \tilde{g} = \mathfrak{O}_1(g) \end{aligned} \tag{15.428}$$

and from (4.92) of Lemma 4.5, (15.426) (with $l_1 + l_2 \le 2$) and (N.594) we get

$$\begin{aligned} \|\mathrm{D}f_{\tilde{g}} - id\|_{\mathbf{D}(0,10\theta_0)\times\mathbf{D}(0,10r_0)} &\lesssim [q^2\rho_{p/q}^5 e^{-qh}\varepsilon_{p/q} + q^4\rho_{p/q}^{-2}e^{-2qh}\varepsilon_{p/q}] \\ &\lesssim (q^2\rho_{p/q}^5 + q^4\rho_{p/q}^{-2}e^{-qh})\varepsilon_{p/q}e^{-qh}. \end{aligned}$$

Because of Lemma 15.4 and (15.428), the Stable Manifold Theorem N.1 of the Appendix shows that the conclusion of the Lemma is true provided for some constant $\mathrm{C} > 0$ (*cf.* (N.589))

$$\|f_{\tilde{g}} - id\|_{\mathrm{C}^1(\mathbf{D}(0,10\theta_0)\times\mathbf{D}(0,10r_0))} \le \mathrm{C}^{-1}\rho_{p/q}\kappa\delta$$

a condition that is implied by (recall (15.427) and the fact that from (15.409) one has $\nu_q \gg q\rho_{p/q}$)

$$(q^2\rho_{p/q}^5 + q^4\rho_{p/q}^{-2}e^{-qh}) < q^2\rho_{p/q}^2 \qquad (< \mathrm{C}^{-1}q^2\rho_{p/q}\nu_q).$$

But (15.405), (15.407) show that this last inequality is satisfied if $q \gg 1$. □

15.5. *Stable and unstable manifolds of* $\Phi_\Omega \circ f_{\rm F}$.

Lemma **15.6.** — *The diffeomorphism* $\Phi_\Omega \circ f_{\rm F}$ *has a hyperbolic* q*-periodic point* $(\overline{\theta}, \overline{r})$ *the local stable and unstable manifolds of which are graphs of* C^1*-functions* $w_-, w_+ :]\overline{\theta} - \rho, \overline{\theta} + \rho[\to \mathbf{R}$ *such that*

$$\begin{cases} (m/2)|\theta - \overline{\theta}| \le |w_+(\theta) - w_-(\theta)| \le 2m|\theta - \overline{\theta}| \quad (\textit{for } \theta \in]\overline{\theta} - \rho, \overline{\theta} + \rho[) \\ m = q(\nu_q e^{-qh}\varepsilon_{p/q}/\varpi)^{1/2}(1 + o_{1/q}(1)) \\ \rho = C^{-1}\nu_q e^{-qh}\varepsilon_{p/q}. \end{cases}$$

Proof. — Recall, *cf.* (15.410), that

$$g_{\rm RNF}^{-1} \circ \Phi_\Omega \circ f_{\rm F} \circ g_{\rm RNF} = \Phi_{2\pi(p/q)r} \circ \Phi_{\overline{\Omega}} \circ f_{\overline{\rm F}^{res}} \circ f_{{\rm F}^{cor}}. \tag{15.429}$$

From (15.417), the pre-image of (θ_1, r_1), by $(\theta, r) \mapsto ([(\theta - \alpha_c)/q]_{\mathrm{mod}\,(2\pi/q)\mathbf{Z}}, \overline{c} + r/q)$ is a q-periodic orbit $O_q \subset \mathbf{T} \times]c_{p/q} - \rho_{p/q}^4, c_{p/q} + \rho_{p/q}^4[\subset \mathbf{T} \times]c_{p/q} - \rho_{p/q}/3, c_{p/q} + \rho_{p/q}/3[$ of $\Phi_{\overline{\Omega}} \circ f_{\overline{\rm F}^{res}}$ as well as of $\Phi_{2\pi(p/q)r} \circ \Phi_{\overline{\Omega}} \circ f_{\overline{\rm F}^{res}}$ ($\overline{\rm F}^{res}$ is $2\pi/q$-periodic); Lemma 15.5 tells us that this periodic orbit is hyperbolic. Let $u_0 \in \mathbf{T} \times]c_{p/q} - \rho_{p/q}^4, c_{p/q} + \rho_{p/q}^4[$ be a point of O_q and denote $\varphi = \Phi_{2\pi(p/q)r} \circ \Phi_{\overline{\Omega}} \circ f_{\overline{\rm F}^{res}}$. One has $\varphi^q(u_0) = u_0$ and we want to find a hyperbolic fixed point for $(\varphi \circ f_{{\rm F}^{cor}})^q$ (the q-th iterate of $\varphi \circ f_{{\rm F}^{cor}}$) close to u_0.

We can write

$$(\varphi \circ f_{{\rm F}^{cor}})^q = \varphi^q \circ j$$

where

$$j = (\varphi^{-(q-1)} \circ f_{{\rm F}^{cor}} \circ \varphi^{q-1}) \circ \cdots \circ (\varphi^{-1} \circ f_{{\rm F}^{cor}} \circ \varphi) \circ f_{{\rm F}^{cor}}.$$

Since $\|(\Phi_{2\pi(p/q)r} \circ \Phi_{\overline{\Omega}})^n\|_{C^2(\mathbf{T} \times]c_{p/q} - \rho_{p/q}/3, c_{p/q} + \rho_{p/q}/3[)} \lesssim 1$ uniformly in n and

$$\|\overline{\rm F}^{res}\|_{C^2(\mathbf{T} \times]c_{p/q} - \rho_{p/q}/3, c_{p/q} + \rho_{p/q}/3[)} \lesssim \varepsilon_{p/q}\rho_{p/q}^{-2} \lesssim 1$$

one has for $n \le \rho_{p/q}^2/\varepsilon_{p/q}$,

$$\|\varphi^n\|_{C^2(\mathbf{T} \times]c_{p/q} - \rho_{p/q}/3, c_{p/q} + \rho_{p/q}/3[)} \lesssim 1 \tag{15.430}$$

and consequently ($q \ll \overline{\varepsilon}^{-1}$)

$$\begin{aligned} \|j - id\|_{C^1(\mathbf{T} \times]c_{p/q} - \rho_{p/q}/3, c_{p/q} + \rho_{p/q}/3[)} &\lesssim q\|{\rm F}^{cor}\|_{C^1(\mathbf{T} \times]c_{p/q} - \rho_{p/q}/3, c_{p/q} + \rho_{p/q}/3[)} \\ &\lesssim \exp(-\rho_{p/q}^{-1/3}) \end{aligned}$$

where we have used (15.411).

Replacing φ^q and j by $\mathrm{T}\circ\varphi^q\circ\mathrm{T}^{-1}$ and $\mathrm{T}\circ j\circ\mathrm{T}^{-1}$ where $\mathrm{T}: u\mapsto u-u_0$ we can assume that $u_0=0\in\mathbf{D}_{\mathbf{R}}(\widetilde{c},\rho_{p/q}^4)^2\subset\mathbf{D}_{\mathbf{R}}(\widetilde{c},\rho_{p/q}/3)^2\subset\mathbf{T}\times]\widetilde{c}-|c_{p/q}/3|,\widetilde{c}+\rho_{p/q}/3[$. We then have $\varphi^q(0)=0$ and the matrix $\mathrm{D}\varphi^q(0)$ is (κ,δ)-hyperbolic with

$$\delta\kappa=q^2\nu_q e^{-qh}\varepsilon_{p/q}(1+o_{1/q}(1)).$$

Write $\varphi^q(u)=\mathrm{D}\varphi^q(0)\xi(u)$ with $\xi(0)=0$, $\mathrm{D}\xi(0)=id$ so that

$$\varphi^q\circ j=\mathrm{D}\varphi^q(0)\circ\xi\circ j.$$

Observe that for $0<\rho<\rho_{p/q}/4$ and $k=0,1$

$$\begin{aligned}\|\mathrm{D}^k(\xi\circ j-id)\|_{\mathrm{C}^0(\mathbf{D}_{\mathbf{R}}(0,\rho))^2}&\lesssim\|\mathrm{D}^k(\xi-id)\|_{\mathrm{C}^0(\mathbf{D}_{\mathbf{R}}(0,\rho))^2}+\|j-id\|_{\mathrm{C}^1(\mathbf{D}_{\mathbf{R}}(0,\rho))^2}\\&\lesssim\rho^{2-k}+q\exp(-\rho_{p/q}^{-1/3}).\end{aligned}$$

Let us choose

$$\rho=\mathrm{C}^{-1}\nu_q e^{-qh}\varepsilon_{p/q}$$

with C large enough. The Stable Manifold Theorem (*cf.* Appendix, Theorem N.1) shows that the diffeomorphism $\varphi^q\circ j$ has a hyperbolic fixed point the stable and unstable manifolds of which are graphs of C^1 functions of the form $\widetilde{w}_-,\widetilde{w}_+:]-\rho/2,\rho/2[\to\mathbf{R}$, $\widetilde{w}_-<0<\widetilde{w}_+$, such that for all $\theta\in]-\rho/2,\rho/2[$ one has

$$(3/2)m_-\theta\le\widetilde{w}_-(\theta)\le(2/3)m_-\theta\le 0\le(2/3)m_+\theta<\widetilde{w}_+(\theta)\le(3/2)m_+\theta.$$

To conclude the proof of the Lemma we set $w_\pm=\widetilde{w}_\pm\circ g_{\mathrm{RNF}}^{-1}$ and note that

$$g_{\mathrm{RNF}}^{-1}\circ\Phi_\Omega\circ f_{\mathrm{F}}\circ g_{\mathrm{RNF}}=\varphi^q\circ j$$

with $\|g_{\mathrm{RNF}}^{-1}-id\|_{\mathrm{C}^1}\le 1/10$ (see (15.412)). □

15.6. *End of the proof of Proposition 15.2.* — Let V be the set

$$\mathrm{V}=\{(\theta,r),\ \theta\in[\overline{\theta},\overline{\theta}+\rho/2],w_-(\theta)\le r\le w_+(\theta)\}$$

the boundary of which is made by two pieces of stable and unstable manifolds and the vertical segment $\mathrm{L}:=\{\overline{\theta}+\rho/2\}\times[w_-(\overline{\theta}+\rho/2),w_+(\overline{\theta}+\rho/2]$. By a theorem of Birkhoff [4] (*cf.* also [21]), any invariant (horizontal) curve of the twist diffeomorphism $\Phi_\Omega\circ f_{\mathrm{F}}$ is the graph of a Lipschitz function $\gamma:\mathbf{T}\to[-1,1]$; if this curve intersects the stable or unstable manifold of $(\overline{\theta},\overline{r})$ it must be included in the union of these stable and unstable manifolds which is impossible. So if this invariant curve intersects the interior of V it has to enter in V by first entering the vertical segment L by the right. But this is clearly impossible also (see Figure 11).

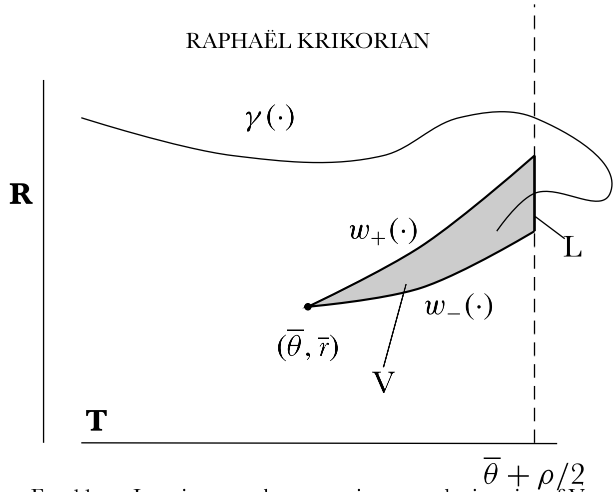

FIG. 11. — Invariant graphs cannot intersect the interior of V

Now the domain V has an area which is

$$\begin{aligned}\text{area}(\mathrm{V}) &\gtrsim \rho \times (m_+ - m_-)\\ &\gtrsim (\nu_q e^{-qh}\varepsilon_{p/q})^{3/2}.\end{aligned}$$

This concludes the proof of Proposition 15.2 if we notice that the dependence on h of the implicit constant in the symbol $\gtrsim$ appears only when we apply Proposition G.1 on Resonant Normal Forms (*cf.* Remark G.1). □

16. Divergent BNF: proof of Theorems E, B and B'

We now use the result of the previous Section to construct examples of real analytic symplectic diffeomorphisms of the disk and the annulus with divergent BNF.

16.1. *Proof of Theorems E and B: the (AA) Case.* — Let $f = \Phi_{2\pi\omega_0 r} \circ f_{\mathrm{O}(r^2)}$ be a real analytic symplectic twist map of the annulus of the form (1.6) and satisfying the twist condition (1.14). We perform a Birkhoff Normal Form, *cf.* Proposition 6.2, on f up to order $\overline{a}_3$, where $\overline{a}_3$ is the integer of Proposition G.1 of the Appendix that appears in Proposition 15.1: there exist $\overline{\rho} > 0$, $g \in \mathrm{Symp}_{ex,\sigma}(\mathbf{T}_h \times \mathbf{D}(0,\overline{\rho}))$ exact symplectic, $\Omega \in \mathcal{O}_\sigma(\mathbf{D}(0,\overline{\rho}))$, $\mathrm{F} \in \mathcal{O}_\sigma(\mathbf{T}_h \times \mathbf{D}(0,\overline{\rho}))$ such that

$$g^{-1} \circ f \circ g = \Phi_\Omega \circ f_{\mathrm{F}}$$

where ($b_2 \neq 0$)

$$(2\pi)^{-1}\Omega(r) = \omega_0 r + b_2 r^2 + \mathrm{O}(r^3), \qquad \mathrm{F}(\theta, r) = \mathrm{O}(r^{\overline{a}_3}), \qquad g - id = \mathrm{O}(r^2).$$

Note that for $\overline{\rho}$ small enough Ω satisfies a $((5/2)\min(b_2, b_2^{-1}), \mathrm{B})$-twist condition on $\mathbf{D}(0,\overline{\rho})$ ($\mathrm{B} \geq 1$). In particular, if $(p_n/q_n)_{n\geq 1}$ are the convergents of ω_0 and $c_n \in \mathbf{R}$ (n large enough) is the point where

$$(2\pi)^{-1}\partial\Omega(c_n) = p_n/q_n, \qquad |c_n| \le \frac{(5/2)\max(b_2(f), b_2(f)^{-1})}{q_n q_{n+1}} \tag{16.431}$$

(*cf.* (15.404) and the twist condition satisfied by Ω) one has

$$\|\mathrm{F}\|_{\mathbf{T}_h\times\mathbf{D}(c_n,|c_n|^2)} \lesssim |c_n|^{\bar a_3}.$$

For $(\zeta_{1,k})_{k\ge 1}, (\zeta_{2,k})_{k\ge 1} \in [-1,1]^{\mathbf{N}^*}$, let $\mathrm{G}_\zeta \in \mathbf{T}_h \times \mathbf{D}(0,1)$ defined by

$$\mathrm{G}_\zeta(\theta, r) = r^{\bar a_3} \sum_{k\ge 1} \zeta_{1,k} e^{-q_k h}\cos(q_k\theta) + \zeta_{2,k} e^{-q_k h}\sin(q_k\theta).$$

We now define $f_\zeta \in \mathrm{Symp}^{\mathcal{O}}_\sigma(\mathbf{T}\times\mathbf{D}(0,1))$, $\mathrm{F}_\zeta \in \mathcal{O}_\sigma(\mathbf{T}_h \times \mathbf{D}(0,\bar\rho))$ by

$$f_\zeta = f \circ f_{\mathrm{G}_\zeta},$$
$$\Phi_\Omega \circ f_{\mathrm{F}_\zeta} := g^{-1}\circ f_\zeta \circ g = \Phi_\Omega \circ f_{\mathrm{F}} \circ g^{-1} \circ f_{\mathrm{G}_\zeta} \circ g.$$

Lemma **16.1.** — *For n, c_n as above, there exists a set $\mathrm{J}_n(\mathrm{F}) \subset [-1,1]^2$ of 2-dimensional Lebesgue measure $\lesssim |c_n|^{1/2}$ such that for any $\zeta \in ([-1,1]^2)^{\mathbf{N}^*}$, such that $\zeta_n \in [-1,1]^2 \smallsetminus \mathrm{J}_n(\mathrm{F})$ one has*

$$m_{\Phi_\Omega\circ f_{\mathrm{F}_\zeta}}(c_n) \gtrsim |c_n|^{2\bar a_3+1} e^{-4q_n h}.$$

Proof. — Let $\alpha_n \in \mathbf{T}$ and $\nu_{q_n} \ge 0$ be such that

$$\widehat{\mathrm{F}}(q_n, c_n)e^{iq_n\theta} + \widehat{\mathrm{F}}(-q_n, c_n)e^{-iq_n\theta} = |c_n|^{\bar a_3}\nu_{q_n} e^{-q_n h}\cos(q_n\theta + \alpha_{q_n}).$$

Since $\Phi_\Omega \circ f_{\mathrm{F}_\zeta} = \Phi_\Omega \circ f_{\mathrm{F}} \circ g^{-1} \circ f_{\mathrm{G}_\zeta} \circ g$, $\mathrm{F}, \mathrm{G}_\zeta = \mathrm{O}(r^{\bar a_3})$ and $g - id = \mathrm{O}(r^2)$ we see that

$$\mathrm{F}_\zeta = \mathrm{F} + \mathrm{G}_\zeta + \mathrm{O}(r^{\bar a_3+1}). \tag{16.432}$$

We now assume $c_n > 0$ for simpler notations (the case $c_n < 0$ is treated in the same way).

We can write

$$\mathrm{G}_\zeta(\theta, r) = r^{\bar a_3}\sum_{k\ge 1} \widetilde{\zeta}_{1,k} e^{-q_k h}\cos(q_k\theta + \alpha_{q_k}) + \widetilde{\zeta}_{2,k} e^{-q_k h}\sin(q_k\theta + \alpha_{q_k})$$

with $\widetilde{\zeta}_{1,k} - i\widetilde{\zeta}_{2,k} = e^{-i\alpha_{q_k}}(\zeta_{1,k} - i\zeta_{2,k})$ and from (16.432) we see that

$$\widehat{\mathrm{F}}_\zeta(q_n, c_n)e^{iq_n\theta} + \widehat{\mathrm{F}}_\zeta(-q_n, c_n)e^{-iq_n\theta}$$
$$= |c_n|^{\bar a_3} e^{-q_n h}\Big((\nu_{q_n} + \widetilde{\zeta}_{1,n} + u_n(\zeta))\cos(q_n\theta + \alpha_{q_n}) + (\widetilde{\zeta}_{2,n} + v_n(\zeta))\sin(q_n\theta + \alpha_{q_n})\Big)$$

where

$$\sup_{\zeta\in([-1,1]^2)^{\mathbf{N}^*}}(|u_n(\zeta)|,|v_n(\zeta)|)\lesssim|c_n|.$$

We can thus write for $\zeta_1,\zeta_2\in]-1,1[^{\mathbf{N}^*}$

$$2|\widehat{\mathrm{F}}_\zeta(q_n,c_n)|=\nu_n(\zeta)|c_n|^{\overline{a}_3}e^{-q_nh}$$

with

$$\begin{aligned}\nu_n(\zeta)^2&=(\nu_{q_n}+\widetilde{\zeta}_{1,n}+u_n(\zeta))^2+(\widetilde{\zeta}_{2,n}+v_n(\zeta))^2\\&\geq|\widetilde{\zeta}_{2,n}-\mathrm{O}(c_n)|^2.\end{aligned}$$

Since $\widetilde{\zeta}_{1,k}-i\widetilde{\zeta}_{2,k}=e^{-i\alpha_{q_k}}(\zeta_{1,k}-i\zeta_{2,k})$, one can hence find a set $\mathrm{J}_n(\mathrm{F})\subset[-1,1]^2$ of 2-dimensional Lebesgue measure

$$|\mathrm{J}_n(\mathrm{F})|\lesssim|c_n|^{1/2}$$

such that

$$(\zeta_{1,n},\zeta_{2,n})\in[-1,1]^2\smallsetminus\mathrm{J}_n(\mathrm{F})\implies|\nu_n(\zeta)|\gtrsim|c_n|^{1/2}.$$

By Proposition 15.1 we thus have

$$m_{\Phi_\Omega\circ f_{\mathrm{F}_\zeta}}(c_n)\gtrsim|c_n|^{2\overline{a}_3+1}e^{-4q_nh}.$$ □

Lemma **16.2.** — *Let* $\mathcal{N}\subset\mathbf{N}$ *be infinite. Then, for almost every* $\zeta\in([-1,1]^2)^{\mathbf{N}^*}$ *for the product measure* $\mu_\infty=(\mathrm{Leb}_{[-1,1]^2})^{\otimes\mathbf{N}^*}$*, there exists an infinite subset* $\widetilde{\mathcal{N}}\subset\mathcal{N}$ *such that for all* $n\in\widetilde{\mathcal{N}}$

$$m_{\Phi_\Omega\circ f_{\mathrm{F}_\zeta}}(c_n)\gtrsim|c_n|^{2\overline{a}_3+1}e^{-4q_nh}.$$

Proof. — Since the random variables ζ_n, $n\in\mathcal{N}$ are independent, for any $m\in\mathbf{N}$, the event $\{\zeta_n\in\mathrm{J}_n(\mathrm{F}),\ \forall\, n\geq m\}$ has zero μ_∞-probability as well as their union. Hence for μ_∞-almost every $\zeta\in\mathcal{X}$, one has for infinitely many $n\in\mathcal{N}$, $\zeta_n\notin\mathrm{J}_n(\mathrm{F})$ and we conclude by Lemma (16.1). □

16.1.1. *Proof of Theorems E and B.* — We now observe that if ω_0 is Diophantine with exponent τ

$$\tau=\limsup\frac{\ln q_{n+1}}{\ln q_n},$$

then for any $\beta>0$ there exists a infinite set $\mathcal{N}_\beta$ such that for all $n\in\mathcal{N}_\beta$

$$q_{n+1}\geq q_n^{\tau-\beta/4}.$$

On the other hand

$$|c_n| \asymp \left|\omega_0 - \frac{p_n}{q_n}\right| \asymp \frac{1}{q_n q_{n+1}} \lesssim \frac{1}{q_n^{1+\tau-\beta/4}}$$

hence

$$q_n \lesssim (1/|c_n|)^{(1/(1+\tau))+\beta/4}$$

and consequently, from Lemma 16.2, for an infinite number of $n \in \mathcal{N}_\beta$

$$m_{\Phi_\Omega \circ f_{F_\zeta}}(c_n) \gtrsim |c_n|^{2\overline{a}_3+1} e^{-4q_n h} \gtrsim \exp\left(-\left(\frac{1}{|c_n|}\right)^{(\frac{1}{1+\tau})+\beta/2}\right). \tag{16.433}$$

We observe that since $t_n \geq 2|c_n|$ (*cf.* (1.20) and (16.431)) one has

$$m_{f_\zeta}(t_n) \gtrsim m_{\Phi_\Omega \circ f_{F_\zeta}}(c_n) \gtrsim \exp\left(-\left(\frac{1}{|t_n|}\right)^{(\frac{1}{1+\tau})+\beta}\right). \tag{16.434}$$

If β is chosen so that

$$\left(\frac{1}{1+\tau}\right) + \beta < \frac{1}{\tau} - \beta,$$

the estimate (16.434), when compared to the conclusion of Theorem A, shows that the Birkhoff Normal Form of f_ζ is divergent for μ_∞-almost every $\zeta \in ([-1,1]^2)^{\mathbf{N}^*}$.

This concludes the proof of Theorem E and, as a consequence, of that of Theorem B (in the (AA) case). □

16.2. *Proof of Theorems E', B and B': (CC) Case.* — Let f be a real analytic symplectic diffeomorphism of the disk admitting the origin 0 as an elliptic equilibrium with irrational frequency ω_0 and satisfying the twist condition (1.14); we *assume* that it is of the form

$$f = \Phi_{\Omega((1/2)(x^2+y^2))} + O((x^2+y^2)^{\overline{a}_3})$$

with $\Omega \in \mathcal{O}_\sigma(\mathbf{D}(0,1))$. Passing to the (z,w)-variables (*cf.* (4.77)) we can write

$$\varphi \circ f \circ \varphi^{-1} = \Phi_\Omega \circ f_F$$

where $F \in \mathcal{O}_\sigma(\mathbf{D}(0,1)^2)$

$$F(z,w) = O((zw)^{\overline{a}_3}) \text{ (and not only } O^{\overline{a}_3}(z,w)).$$

Let as before $(p_n/q_n)_{n\geq 1}$ be the convergents of ω_0 and $c_n \in \mathbf{R}$ the point where $(2\pi)^{-1}\partial\Omega(c_n) = p_n/q_n$ (*cf.* (15.404)).

For $(\zeta_n)_{n\in\mathbf{N}^*}\in([-1,1]^2)^{\mathbf{N}^*}$, let $G_\zeta\in\mathcal{O}_\sigma(\mathbf{D}(0,1)^2)$

$$G_\zeta(z,w)=(-izw)^{\overline{a}_3}\sum_{k=1}^{\infty}\frac{\zeta_{1,k}}{2}\times((i^{-1/2}z)^{q_k}+(i^{-1/2}w)^{q_k})$$
$$+\frac{\zeta_{2,k}}{2i}\times((i^{-1/2}z)^{q_k}-(i^{-1/2}w)^{q_k}).$$

We now define (recall the definition of G_ζ^* in Section 1.4.3)

$$\Phi_\Omega\circ f_{F_\zeta}=\Phi_\Omega\circ f_F\circ\Phi_{G_\zeta}$$
$$f_\zeta=\varphi^{-1}\circ(\Phi_\Omega\circ f_{F_\zeta})\circ\varphi=f\circ\Phi_{G_\zeta\circ\varphi}=f\circ\Phi_{G_\zeta^*}.\qquad (cf.\ (4.82))$$

Lemma **16.3.** — *Assume that for some n large enough, c_n is positive. Then, there exists $J_n(F)\subset[-1,1]^2$ of Lebesgue measure $\lesssim c_n^{1/2}$ such that if $\zeta_n=(\zeta_{1,n},\zeta_{2,n})\notin J_n(F)$ one has*

$$m_{f_\zeta}(2c_n)\gtrsim m_{\Phi_\Omega\circ f_{F_\zeta}}(c_n)\gtrsim c_n^{2\overline{a}_3+1}e^{-4q_nh}.$$

Proof. — We define

$$h_n=-(1/2)\ln(c_n+c_n^2),$$

and since $|\omega_0-\frac{p_n}{q_n}|\asymp\frac{1}{q_nq_{n+1}}\asymp c_n$ one has

$$h_n=(-1/2)\ln c_n+O(c_n)$$
$$=(-1/2)\ln c_n-O(1/q_n^2)$$

hence

$$(\mathbf{16.435})\qquad e^{-q_nh_n}=c_n^{q_n/2}e^{O(1/q_n)}<c_n^{10}.$$

Let $W_n^{CC}=W^{CC}_{h_n,\mathbf{D}(c_n,c_n^2)}=\{(z,w)\in\mathbf{C}^2,\ \max(|z|,|w|)\le e^{h_n}(c_n+c_n^2)^{1/2},\ -izw\in\mathbf{D}(c_n,c_n^2)\}$. One has

$$(\mathbf{16.436})\qquad \|F\|_{W_n^{CC}}\lesssim|c_n|^{\overline{a}_3},\qquad \|G_\zeta\|_{W_n^{CC}}\lesssim c_n^{\overline{a}_3}.$$

Using Lemma K.1 we can pass to (AA)-coordinates: if ψ_- is the diffeomorphism defined in (4.79)

$$\psi_-^{-1}(W^{CC}_{h_n,\mathbf{D}(c_n,c_n^2)})\supset W^{AA}_{h_n,\mathbf{D}(c_n,c_n^2)}=\mathbf{T}_{h_n}\times\mathbf{D}(c_n,c_n^2)$$

and we can introduce F^{AA}, $F^{AA}_\zeta\in\mathcal{O}_\sigma(\mathbf{T}_{h_n}\times\mathbf{D}(c_n,c_n^2))$ (*cf.* (4.82))

$$\psi_-^{-1}\circ f_F\circ\psi_-=f_{F^{AA}},\qquad \psi_-^{-1}\circ f_{F_\zeta}\circ\psi_-=f_{F^{AA}_\zeta}=f_{F^{AA}}\circ\Phi_{G_\zeta\circ\psi_-}.$$

Since $F^{AA} = F \circ \psi_- + \mathfrak{O}_2(F)$ and $F^{AA}_\zeta = F^{AA} + G_\zeta \circ \psi_- + \mathfrak{O}_2(F^{AA}, G_\zeta \circ \psi_-)$ (*cf.* (4.94), (4.92)) one has on W^{AA}_n

$$\|F^{AA}_\zeta\|_{W^{AA}_n} \lesssim c_n^{\bar{a}_3}, \qquad F^{AA}_\zeta = F^{AA} + G_\zeta \circ \psi_- + O(c_n^{(3/2)\bar{a}_3}). \tag{16.437}$$

If we define ν_n and $\alpha_n \in \mathbf{T}$ by

$$\widehat{F}^{AA}(q_n, c_n)e^{iq_n\theta} + \widehat{F}^{AA}(-q_n, c_n)e^{-iq_n\theta} = |c_n|^{\bar{a}_3}\nu_{q_n}e^{-q_nh_n}\cos(q_n\theta + \alpha_{q_n})$$

we see that on $\mathbf{T}_{h_n-1} \times \mathbf{D}(c_n, c_n^2/2)$ (*cf.* (16.437))

$$F^{AA}_\zeta = F^{AA}(\theta, r) + r^{\bar{a}_3}\sum_{k=1}^{\infty} r^{q_k/2}(\zeta_{1,k}\cos(q_k\theta) + \zeta_{2,k}\sin(q_k\theta)) + O(c_n^{(3/2)\bar{a}_3}).$$

Hence

$$\begin{aligned}
&\widehat{F}^{AA}_\zeta(q_n, c_n)e^{iq_n\theta} + \widehat{F}_\zeta(-q_n, c_n)e^{-iq_n\theta} \\
&\quad = c_n^{\bar{a}_3}\Bigg(\bigg((\nu_{q_n} + O(c_n^{(1/2)\bar{a}_3}))e^{-q_nh_n} + c_n^{q_n/2}\widetilde{\zeta}_{1,n}\bigg)\cos(q_n\theta + \alpha_{q_n}) \\
&\qquad + \bigg(c_n^{q_n/2}\widetilde{\zeta}_{2,n} + O(c_n^{(1/2)\bar{a}_3})e^{-q_nh_n}\bigg)\sin(q_n\theta + \alpha_{q_n})\Bigg)
\end{aligned}$$

with $\widetilde{\zeta}_{1,k} - i\widetilde{\zeta}_{2,k} = e^{-i\alpha_{q_k}}(\zeta_{1,k} - i\zeta_{2,k})$. We thus have (*cf.* (16.435))

$$\begin{aligned}
2|\widehat{F}^{AA}_\zeta(q_n, c_n)| &\geq c_n^{\bar{a}_3}|c_n^{q_n/2}\widetilde{\zeta}_{2,n} + O(c_n^{(1/2)\bar{a}_3})e^{-q_nh_n}| \\
&\geq c_n^{\bar{a}_3}|e^{-q_nh_n}e^{O(1/q_n)}\widetilde{\zeta}_{2,n} + O(c_n^{(1/2)\bar{a}_3})e^{-q_nh_n}| \\
&\gtrsim c_n^{\bar{a}_3}e^{-q_nh_n}|\widetilde{\zeta}_{2,n} + O(c_n^{(1/2)\bar{a}_3})|
\end{aligned}$$

and we see that if

$$|\widetilde{\zeta}_{2,n}| \geq c_n^{1/2}$$

one can apply Proposition 15.1 (*cf.* (16.435)):

$$m_{\Phi_\Omega \circ f_{F^{AA}_\zeta}}(c_n) \gtrsim C_{h_n}^{-1}c_n^{2\bar{a}_3+1}e^{-4q_nh} \gtrsim c_n^{2\bar{a}_3+1}e^{-4q_nh}.$$

Now, since c_n *is positive* and $m_{f_\zeta}(2c_n) \gtrsim m_{\Phi_\Omega \circ F_\zeta}(c_n)$ this provides

$$m_{f_\zeta}(2c_n) \gtrsim m_{\Phi_\Omega \circ f_F \circ \Phi_{G_\zeta}}(c_n) \gtrsim c_n^{2\bar{a}_3+1}e^{-4q_nh}. \qquad \square$$

We can deduce the analogue of Lemma 16.2

Lemma **16.4.** — *Let $\mathcal{N}$ be an infinite set of $n \in \mathbf{N}$ for which $c_n > 0$. Then, for almost every $\zeta \in ([-1,1]^2)^{\mathbf{N}^*}$, there exists an infinite subset $\widetilde{\mathcal{N}} \subset \mathcal{N}$ such that for all $n \in \widetilde{\mathcal{N}}$*

$$m_{f_\zeta}(2c_n) \gtrsim m_{\Phi_\Omega \circ f_{F_\zeta}}(c_n) \gtrsim c_n^{2\overline{a}_3+1} e^{-4q_n h}.$$

16.2.1. *Proof of Theorems E' and B (CC) Case, ω_0 Diophantine.* — We want to apply the previous Lemma 16.4 to an infinite set $\mathcal{N}$ such that for all $n \in \mathcal{N}$ one has both

$$(\mathbf{16.438}) \qquad c_n > 0 \quad \text{and} \quad q_{n+1} \geq q_n^{\tau-}.$$

Such a set may not exist for arbitrary choices of ω_0 (Diophantine) and Ω. On the other hand, if one chooses the sign of $\partial^2\Omega(0)$ depending on ω_0 (or more precisely its sequence of convergents) this is possible.

Let $\beta > 0$ and

$$\mathcal{N}_\beta = \{n \in \mathbf{N},\ q_{n+1} \geq q_n^{\tau-\beta/2}\}, \qquad \mathcal{Q}_\beta = \{p_n/q_n,\ n \in \mathcal{N}_\beta\}.$$

Since $\mathcal{N}_\beta$ is infinite, one of the two sets $\mathcal{Q}_\beta^{\pm} = \mathcal{Q}_\beta \cap (]\omega_0, \pm\infty[)$ is infinite. We define $s_\beta(\omega_0)$ as the non-empty subset of $\{-1,1\}$ such that $\pm 1 \in s_\beta(\omega_0)$ if and only if $\mathcal{Q}_\beta^{\pm}$ is infinite.

We now assume that $\omega(r) := (2\pi)^{-1}\partial\Omega(r)$ is of the form

$$\omega(r) = \omega_0 + 2b_2 r + O(r^2), \qquad \text{with} \quad \operatorname{sign}(b_2) \in s_\beta(\omega_0).$$

For the sake of simplicity we shall assume that $1 \in s(\omega_0)$ and $b_2 > 0$ (the case $-1 \in s(\omega_0)$ and $b_2 < 0$ is treated similarly).

The sets

$$\mathcal{N}'_\beta := \mathcal{N}_\beta^+ = \{n \in \mathcal{N}_\beta,\ p_n/q_n > \omega_0\}, \qquad \mathcal{C}_\beta = \omega^{-1}(\mathcal{Q}_\beta^+)$$

are then infinite; note that $\mathcal{C}_\beta \subset]0,\infty[$ and its points c_n, $n \in \mathcal{N}_\beta^+$ accumulate zero.

We then apply Lemma 16.4: for almost every ζ and infinitely many $n \in \mathcal{N}_\beta^+$

$$m_{f_\zeta}(2c_n) \gtrsim c_n^{2\overline{a}_3+1} e^{-4q_n h}$$

and, arguing like in Subsection 16.1.1, we see that setting $t_n = 2c_n$ we have for infinitely many $n \in \mathcal{N}'_\beta = \mathcal{N}_\beta^+$

$$m_{f_\zeta}(t_n) \gtrsim \exp\left(-\left(\frac{1}{|t_n|}\right)^{(\frac{1}{1+\tau})+\beta}\right). \qquad \square$$

16.2.2. *Proof of Theorems E' and B': (CC) Case, ω_0 Liouvillian.* — Since ω_0 is Liouvillian, there exists an infinite set $\mathcal{N} \subset \mathbf{N}$ such that

$$\lim_{n\in\mathcal{N}} \frac{\ln q_{n+1}}{\ln q_n} = \infty.$$

We define

$$\mathcal{Q}^{\pm} = \{p_n/q_n,\ n \in \mathcal{N},\ p_n/q_n \in]\omega_0, \pm\infty[\}$$

and $s(\omega_0)$ as the non-empty subset of $\{-1, 1\}$ (one of the two sets $\mathcal{Q}^+$, $\mathcal{Q}^-$ is infinite) such that $\pm 1 \in s(\omega_0)$ if and only if $\mathcal{Q}^{\pm}$ is infinite.

We assume that $1 \in s(\omega_0)$ and $b_2 > 0$ (the case $-1 \in s(\omega_0)$ and $b_2 < 0$ is treated similarly) and we set $\mathcal{N}^+ = \{n \in \mathcal{N},\ p_n/q_n > \omega_0\}$. The set $\mathcal{C} = \omega^{-1}(\mathcal{Q}^+)$ is infinite, contained in $]0, \infty[$ and its points c_n, $n \in \mathcal{N}^+$ accumulate 0.

We now apply Lemma 16.4: for almost every ζ one has for infinitely many $n \in \mathcal{N}^+$

$$m_{f_\zeta}(2c_n) \gtrsim c_n^{2\overline{a}_3+1} e^{-4q_n h}.$$

For these n's one has

$$c_n \asymp |\omega_0 - \frac{p_n}{q_n}| \asymp \frac{1}{q_n q_{n+1}},$$

and, for any $\beta > 0$, provided n is large enough,

$$c_n^{2\overline{a}_3+1} e^{-4q_n h} \gtrsim (\frac{1}{q_{n+1}})^{2(2\overline{a}_3+1)} \exp(-q_{n+1}^{\beta/2}) \gtrsim \exp(-q_{n+1}^{\beta}).$$

If we set $t_n = 2c_n$ (*cf.* (16.431))

$$2c_n \le t_n := \frac{5(b_2 + b_2^{-1})}{q_n q_{n+1}}$$

hence

$$m_{f_\zeta}(t_n) \ge m_{f_\zeta}(2c_n) \gtrsim \exp(-q_{n+1}^{\beta})$$

for infinitely many n in $\mathcal{N}' := \mathcal{N}^+$. □

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Appendix A: Estimates on composition and inversion

A.1 *Proof of Lemma 4.4.* — We shall do the proof in the (AA)-case; the proof in the (CC)-case follows the same lines.

We can assume that $e^{-\delta}W_{h,U} = W_{h-\delta/2,e^{-\delta}U}$ is not empty (otherwise, there is nothing to prove).

By (2.53), there exists a numerical constant $C > 0$ such that if $\delta > 0$ satisfies

$$C\|F\|_{W_{h,U}}\delta^{-2}\underline{d}(W_{h,U})^{-1} < 1, \tag{A.439}$$

then for any fixed $\theta \in \mathbf{T}_{h-2\delta}$ and any fixed $r \in e^{-\delta}U$, the map $U \to U$, $R \mapsto r - \partial_\theta F(\theta, R)$ is contracting and by the Contraction Mapping Principle there thus exists a unique $R \in U$ depending holomorphically on $(\theta, r) \in \mathbf{T}_{h-\delta/2} \times e^{-\delta}U$ such that

$$r = R + \partial_\theta F(\theta, R).$$

On the other hand, assumption (A.439) and Cauchy's inequality (2.53) show that if C is large enough

$$|\partial_R F(\theta, R)| \lesssim \delta^{-1} \times \|F\|_{h,U} < (1/2)\delta,$$

hence

$$\varphi := \theta + \partial_R F(\theta, R) \in \mathbf{T}_{h-\delta}.$$

We can thus define a holomorphic map

$$f_{\mathrm{F}}:\mathbf{T}_{h-\delta/2}\times e^{-\delta}\mathrm{U}\to\mathbf{T}_{h-\delta}\times\mathrm{U}$$

by

$$f_{\mathrm{F}}(\theta,r)=(\varphi,\mathrm{R})\iff\begin{cases}r=\mathrm{R}+\partial_\theta\mathrm{F}(\theta,\mathrm{R})\\ \varphi=\theta+\partial_{\mathrm{R}}\mathrm{F}(\theta,\mathrm{R}).\end{cases}\tag{A.440}$$

Notice that the maps $(\theta,r)\mapsto\varphi(\theta,r)-\theta$, $(\theta,r)\mapsto\mathrm{R}(\theta,r)-r$ such defined are Lipschitz with Lipschitz constant $\lesssim\delta^{-2}\,\underline{\mathrm{d}}(\mathrm{U})^{-2}\|\mathrm{F}\|_{h,\mathrm{U}}$. Thus, if for some numerical constant large enough

$$\mathrm{C}\delta^{-2}\,\underline{\mathrm{d}}(\mathrm{U})^{-2}\|\mathrm{F}\|_{h,\mathrm{U}}<1,\tag{A.441}$$

the map f_{F} is a holomorphic diffeomorphism from $\mathbf{T}_{h-\delta/2}\times e^{-\delta}\mathrm{U}$ onto its image.

Conversely, if (A.439) is satisfied, given $(\varphi,\mathrm{R})\in\mathbf{T}_{h-\delta}\times e^{-2\delta}\mathrm{U}$, the same arguments as those developed above show there exists a unique $(\theta,r)\in\mathbf{T}_{h-\delta/2}\times e^{-\delta}\mathrm{U}$ such that $f(\theta,r)=(\varphi,\mathrm{R})$. We thus have if (A.441) is satisfied

$$\mathbf{T}_{h-\delta}\times e^{-2\delta}\mathrm{U}\subset f_{\mathrm{F}}(\mathbf{T}_{h-\delta/2}\times e^{-\delta}\mathrm{U}).$$

Finally, we observe that the diffeomorphism f_{F} is *exact symplectic* which means that the differential form $\mathrm{R}d\varphi-rd\theta$ is exact; in particular, it is symplectic. Indeed

$$\begin{aligned}\mathrm{R}d\varphi-rd\theta&=-\varphi d\mathrm{R}+d(\varphi\mathrm{R})-rd\theta\\&=-(\theta+\partial_{\mathrm{R}}\mathrm{F}(\theta,\mathrm{R}))d\mathrm{R}-(\mathrm{R}+\partial_\theta\mathrm{F}(\theta,\mathrm{R}))d\theta+d(\varphi\mathrm{R})\\&=-d\mathrm{F}+d(\varphi\mathrm{R})-d(\theta\mathrm{R})\\&=d(-\mathrm{F}+(\varphi-\theta)\mathrm{R})\end{aligned}$$

(observe that the function $-\mathrm{F}+(\varphi-\theta)\mathrm{R}=-\mathrm{F}(\theta,\mathrm{R})+\partial_{\mathrm{R}}\mathrm{F}(\theta,\mathrm{R})\mathrm{R}$ is well defined on $\mathbf{T}_h\times\mathrm{U}$). We have thus proven that there exists a numerical constant $\overline{\mathrm{C}}>0$ such that if

$$\overline{\mathrm{C}}\delta^{-2}\,\underline{\mathrm{d}}(\mathrm{U})^{-2}\|\mathrm{F}\|_{h,\mathrm{U}}<1\tag{A.442}$$

the diffeomorphism f_{F} previously defined is exact symplectic and

$$e^{-2\delta}\mathrm{W}_{h,\mathrm{U}}\subset f_{\mathrm{F}}(e^{-\delta}\mathrm{W}_{h,\mathrm{U}})\subset\mathrm{W}_{h,\mathrm{U}}.\tag{A.443}$$

Estimate (4.91) comes from (A.440) and

$$\max_{i=1,2}|\partial_i\mathrm{F}(\theta,\mathrm{R})-\partial_i\mathrm{F}(\varphi,r)||\le2\|\mathrm{D}^2\mathrm{F}\|\,\|\mathrm{DF}\|.\qquad\square$$

A.2 *Proof of Lemma 4.5.* — We illustrate the proof in the (AA)-Case (it is the same in the (CC)-case).

Since f is close to the identity, the map $\widetilde{f}:(\theta,R)\mapsto(\varphi,r) \iff f(\theta,r)=(\varphi,R)$ defines a diffeomorphism such that $\widetilde{f}-id=\mathfrak{O}(f-id)$ and since f is exact symplectic we know (*cf.* Section A.1) that $\varphi dR+rd\theta=dF$ for some holomorphic function $F:(\theta,R)\mapsto F(\theta,R)$. Since $F(\theta,R)=\int_{\gamma_{\theta,R}}(\varphi dR+rd\theta)$ where $\gamma_{\theta,R}$ is a path joining $(0,R_0)\in\{\theta\in\mathbf{C},|\Im\theta|<h\}\times U$ to (θ,R), the function F, which is unique up to the addition of a constant, thus satisfies $F=\mathfrak{O}(f-id)$.

The estimate (4.92) is a consequence of (4.87) and the fact that

$$|\partial_\theta F(\theta,R)-\partial_\theta F(\theta,r)|\le\|D\partial_\theta F\||R-r| \;\le\|D\partial_\theta F\|\|\partial_\theta F\|$$

$$|\partial_R F(\theta,R)-\partial_R F(\theta,r)|\le\|D\partial_R F\||\varphi-\theta| \;\le\|D\partial_R F\|\|\partial_R F\|. \qquad \square$$

A.3 *Proof of Lemma 4.6.* — *1) Proof of (4.95)*. One has

$$f_F(\theta,r)=(\varphi,R) \iff \begin{cases} r=R+\partial_\theta F(\theta,R)\\ \varphi=\theta+\partial_R F(\theta,R)\end{cases}$$

$$f_\Omega(\varphi,R)=(\psi,Q) \iff \begin{cases} R=Q\\ \psi=\varphi+\partial_Q\Omega(Q)\end{cases}$$

hence $Q=R$ and

$$\begin{aligned}\psi&=\varphi+\partial_R\Omega(R)\\ &=\theta+\partial_R F(\theta,R)+\partial_R\Omega(R)\\ &=\theta+\partial_Q(\Omega+F)(\theta,Q)\end{aligned}$$

thus, since $r=R+\partial_\theta F(\theta,R)=Q+\partial_\theta F(\theta,Q)$ and Ω does not depend on the θ-variable, one has

$$\begin{cases} r=Q+\partial_\theta(\Omega+F)(\theta,Q)\\ \psi=\theta+\partial_Q(\Omega+F)(\theta,Q)\end{cases}$$

which is equivalent to

$$f_{\Omega+F}(\theta,r)=(\psi,Q)=f_\Omega\circ f_F(\theta,r).$$

2) Proof of (4.93). Assume that $f_F(\theta,r)=(\varphi,R)$ and $f_G(\varphi,R)=(\psi,Q)$. Then

$$f_F(\theta,r)=(\varphi,R), \qquad \begin{cases} r=R+\partial_\theta F(\theta,R)\\ \varphi=\theta+\partial_R F(\theta,R)\end{cases} \tag{A.444}$$

$$(\mathbf{A.445})\qquad f_G(\varphi,R)=(\psi,Q),\qquad \begin{cases} R=Q+\partial_\varphi G(\varphi,Q)\\ \psi=\varphi+\partial_Q G(\varphi,Q)\end{cases}$$

$$\begin{aligned}Qd\psi-rd\theta&=Qd\psi-Rd\varphi+Rd\varphi-rd\theta\\&=d(-F-G+(\varphi-\theta)R+(\psi-\varphi)Q).\end{aligned}$$

If $f_G\circ f_F=f_H$ then one has $Qd\psi-rd\theta=d(-H+Q(\psi-\theta))$ and then

$$\begin{aligned}0&=d(-H+F+G+Q(\psi-\theta)-R(\varphi-\theta)-Q(\psi-\varphi))\\&=d(-H+F+G-(Q-R)(\varphi-\theta))\end{aligned}$$

and so

$$H(\theta,Q)=\mathrm{cst}+F(\theta,R)+G(\varphi,Q)-(Q-R)(\varphi-\theta).$$

Let us write $H(\theta,Q)=F(\theta,Q)+G(\theta,Q)+A(\theta,Q)$ where

$$\begin{aligned}-A&=F(\theta,Q)-F(\theta,R)+G(\theta,Q)-G(\varphi,Q)+(Q-R)(\varphi-\theta)\\&=F(\theta,Q)-F(\theta,R)+G(\theta,Q)-G(\varphi,Q)-\partial_\varphi G(\varphi,Q)\partial_R F(\theta,R)\end{aligned}$$

We can now estimate

$$\begin{aligned}\|A\|_{h-\delta,U_\delta}&\le\|\partial_R F\|_{h,U}\|Q-R\|_{h,U}+\|\partial_\varphi G\|_{h,U}\|\varphi-\theta\|_{h,U}\\&\quad+\|\partial_\varphi G(\varphi,Q)\|_{h,U}\|\partial_R F(\theta,R)\|_{h,U}\\&\le\|\partial_R F\|_{h,U}\|\partial_\varphi G\|_{h,U}+\|\partial_\varphi G\|_{h,U}\|\partial_R F\|\\&\quad+\|\partial_\varphi G\|_{h,U}\|\partial_R F\|_{h,U}\end{aligned}$$

and deduce (4.93).

3) Proof of (4.94). We just write

$$\begin{aligned}f_{F+G}\circ f_G^{-1}&=f_{F+G}\circ f_{-G+O(|D^2G|DG|)}\\&=f_{F+\|DF\|\mathcal{O}_1(G)}\qquad(\text{using }(4.93))\end{aligned}$$

and a similar expression for $f_F^{-1}\circ f_{F+G}=f_{-F+O(|D^2F||DF|)}$.

The proof of (4.93) and (4.94) is the same in the (CC)-case. □

A.4 *Proof of Proposition 4.7.* — We first state two lemmata.

Lemma **A.1.** — *Let* W *be an open subset of* $M=\mathbf{C}^2$ *or* $\mathbf{T}_\infty\times\mathbf{C}$, $v\in\mathcal{O}(W)$ *and* $g-id\in\mathcal{O}(W)$ *such that* $\|g-id\|_W\lesssim 1$. *Then if* $\|v\|_W$ *is small enough*

$$(\mathbf{A.446})\qquad (id+v)\circ g\circ(id+v)^{-1}=g\circ(id+[g]\cdot v+\mathfrak{O}_2(v))$$

where

$$[g]\cdot v = -v + (Dg^{-1}\cdot v)\circ g. \tag{A.447}$$

Proof. — One has

$$(id+v)\circ g\circ(id+v)^{-1} = g\circ(id - v + \mathfrak{O}_2(v)) + v\circ g\circ(id - v + \mathfrak{O}_2(v))$$
$$= g - Dg\cdot v + v\circ g + \mathfrak{O}_2(v)$$

hence

$$g^{-1}\circ(id+v)\circ g\circ(id+v)^{-1}$$
$$= g^{-1}\circ\Big(g - Dg\cdot v + v\circ g + \mathfrak{O}_2(v)\Big)$$
$$= id - Dg^{-1}\circ g\cdot Dg\cdot v + Dg^{-1}\circ g\cdot v\circ g + \mathfrak{O}_2(v)$$
$$= id - v + (Dg^{-1}\cdot v)\circ g + \mathfrak{O}_2(v).$$

□

Lemma **A.2.** — *If* $\Omega\in\mathcal{O}(U)$, $Y\in\mathcal{O}(W_{h,U}\cup\Phi_\Omega(W_{h,U}))$ *then*

$$f_Y\circ\Phi_\Omega\circ f_Y^{-1} = \Phi_\Omega\circ f_{[\Omega]\cdot Y+\mathfrak{O}_2(Y)} \tag{A.448}$$

where

$$[\Omega]\cdot Y = Y\circ\Phi_\Omega - Y.$$

Proof. — From Lemma A.1, and (4.92) we have

$$f_Y\circ\Phi_\Omega\circ f_Y^{-1} = \Phi_\Omega\circ(id + [\Phi_\Omega]\cdot(J\nabla Y) + \mathfrak{O}_2(Y)) \tag{A.449}$$
$$= \Phi_\Omega\circ\Big(id - J\nabla Y + (D\Phi_\Omega^{-1}\cdot(J\nabla Y))\circ\Phi_\Omega + \mathfrak{O}_2(Y)\Big).$$

On the other hand

$$J\nabla(Y\circ\Phi_\Omega) = J\,({}^tD\Phi_\Omega)\cdot(\nabla Y\circ\Phi_\Omega). \tag{A.450}$$

Because Φ_Ω is symplectic, one has $J\,{}^t(D\Phi_\Omega) = (D\Phi_\Omega)^{-1}J$, and we deduce from both (A.449), (A.450) that

$$f_Y\circ\Phi_\Omega\circ f_Y^{-1} = \Phi_\Omega\circ(id + J\nabla Y - J\nabla(Y\circ\Phi_\Omega) + \mathfrak{O}_2(Y))$$
$$= \Phi_\Omega\circ f_{[\Omega]\cdot Y+\mathfrak{O}_2(Y)}$$

where

$$[\Omega]\cdot Y = Y\circ\Phi_\Omega - Y.$$

□

From (4.92), (4.93) (we use the fact that $D(O(\|DF\|G)) = \|DF\|\mathfrak{O}_1(G))$

$$f_Y \circ f_F \circ f_Y^{-1} = f_{F+\|DF\|\mathfrak{O}_1(Y)} \tag{A.451}$$

and on the other hand, from Lemma A.2

$$f_Y \circ \Phi_\Omega \circ f_Y^{-1} = \Phi_\Omega \circ f_{[\Omega]\cdot Y + \mathfrak{O}_2(Y)}. \tag{A.452}$$

Hence (we use (4.93) in the last line of the following equations)

$$f_Y \circ \Phi_\Omega \circ f_F \circ f_Y^{-1} = f_Y \circ \Phi_\Omega \circ f_Y^{-1} \circ f_Y \circ f_F \circ f_Y^{-1} \tag{A.453}$$

$$= \Phi_\Omega \circ f_{[\Omega]\cdot Y + \mathfrak{O}_2(Y)} \circ f_{F+\|DF\|\mathfrak{O}_1(Y)} \tag{A.454}$$

$$= \Phi_\Omega \circ f_{F+[\Omega]\cdot Y + \|DF\|\mathfrak{O}_1(Y)}, \tag{A.455}$$

and the conclusion follows if $\widetilde{F} = F + [\Omega]\cdot Y + \|DF\|\mathfrak{O}_1(Y)$. □

Appendix B: Whitney type extensions

B.1 *Proof of Lemma 2.2.* — Let $\chi_\delta : \mathbf{R} \to [0,1]$ be a smooth function with support in $[-1,1]$ and equal to 1 on $[-e^{-\delta/2}, e^{-\delta/2}]$ such that

$$\sup_{\mathbf{R}} |\partial^j \chi_\delta| \lesssim \delta^{-j}. \tag{B.456}$$

We define for $r \in \mathbf{C}$, $\eta(r) = \chi_\delta((e^{\delta/2}|r|)^2/\rho^2)$ and for $i \in J_U$, $\eta_i(r) = (1 - \chi_\delta((e^{-\delta}|r - c_i|)^2/\rho_i^2))$. Note that η is equal to 1 on $e^{-\delta}\mathbf{D}(0,\rho)$ and 0 on $\mathbf{C} \setminus e^{-\delta/2}\mathbf{D}(0,\rho)$ and η_i is equal to 1 on $\mathbf{C} \setminus e^{\delta}\mathbf{D}(c_i,\rho_i)$ and 0 on $e^{\delta/2}\mathbf{D}(c_i,\rho_i)$ hence $\zeta = \eta \prod_{i\in J_U} \eta_i$ is equal to 1 on $e^{-\delta}U$ and 0 on $V := (\mathbf{C} \setminus e^{-\delta/2}\mathbf{D}(0,\rho)) \cup \bigcup_{i\in J_U} e^{\delta/2}\mathbf{D}(c_i,\rho_i)$. The union of the open sets $W_{h,e^{-\delta/10}U}$ (resp. $e^{-1/10}U$) and $W_{h,V}$ (resp. V) is $W_{h,\mathbf{C}}$ (resp. $\mathbf{C}$) and on their intersection the functions ζF and 0 coincide. As a consequence, one can extend ζF by 0 on $W_{h,V}$ as a smooth function $F^{W_h} : W_{h,\mathbf{C}} \to W_{h,\mathbf{C}}$. Note that since ζ is σ-symmetric, the same is true for F^{W_h} and that F^{W_h} and F coincide on $W_{h,e^{-\delta}U}$ (which contains $e^{-\delta}W_{h,U}$).

To get the estimates on the derivatives of F^{W_h} we observe from (B.456) and the definitions of η, η_i that

$$\max_i(\max_{0\le j\le k} \sup_{\mathbf{C}} |D^j\eta|, \max_{0\le j\le k} \sup_{\mathbf{C}} |D^j\eta_i|) \lesssim \delta^{-k} \max_i(\rho^{-2k}, \rho_i^{-2k})$$

and since $\max_i(\eta,\eta_i) \le 1$, one has by Leibniz formula

$$\max_{0\le j\le k} \sup_{\mathbf{C}} |D^j\zeta| \lesssim (\#J_U + 1)^k \delta^{-k} \max_i(\rho^{-2k}, \rho_i^{-2k}).$$

Hence $F^{W_h} := \zeta F$ satisfies

$$\sup_{0\le j\le k} \|D^j F^{W_h}\|_{W_{h,\mathbf{C}}} \le C(1+\#J_U)^k (\delta\, \underline{d}(U))^{-2k} \max_{0\le j\le k} \|D^j F\|_{W_{h,e^{-\delta/10}U}}.$$ □

B.2 *Proof of Lemma 2.5.* — Write $(2\pi)^{-1}\Omega(z) = \sum_{n=0}^{\infty} b_n z^n$ with $|b_n| \le \rho_0^{-n}$, $(2\pi)^{-1}\Omega_2(z) = b_0 + b_1 z + b_2 z^2$, $(2\pi)^{-1}\Omega_{\ge 3}(z) = \sum_{n=3}^{\infty} b_n z^n$. For $0 \le j \le 3$ and $\delta > 0$, there exists $C_j > 0$ such that for any $\rho \le \rho_0/2$

$$\|D^j \Omega_{\ge 3}\|_{\mathbf{D}(0,\rho)} \le C_j \rho^{3-j}. \tag{B.457}$$

Let $\chi : \mathbf{C} \to [0,1]$ be a real symmetric smooth function with support in $\mathbf{D}(0,1)$ and equal to 1 on $\overline{\mathbf{D}}(0,1/2)$. We define the real symmetric function defined on $\mathbf{C}$

$$\Omega_\rho^{Wh}(z) = \Omega_2(z) + \chi(z/\rho)\Omega_{\ge 3}(z).$$

For any $z \in \mathbf{D}(0,\rho/2)$ one has $\Omega_\rho^{Wh}(z) = \Omega(z)$ and by (B.457) and Leibniz formula, for some constant B depending only on b_0, b_1, b_2, $\|D^j\chi\|_{C^0}$, $\|D^j\Omega_{\ge 3}\|_{\mathbf{D}(0,\rho_0)}$, $0 \le j \le 3$, one has

$$\forall z \in \mathbf{C}, \qquad \left|\frac{1}{2\pi} D^3 \Omega_\rho^{Wh}(z)\right| \le B.$$

On the other hand, for some constant C depending only on, $\|\partial^j\chi\|_{C^0}$, $\|\partial^j\Omega\|_{\mathbf{D}(0,\rho_0)}$, $0 \le j \le 2$

$$\forall\, t \in \mathbf{R}, \qquad \left|\frac{1}{2\pi}\partial^2\Omega_\rho^{Wh}(t) - 2b_2\right| \le C\rho$$

and if $\rho = \overline{\rho}$ is chosen small enough so that $C\overline{\rho} < b_2$, one has (we assume $b_2 > 0$) $b_2 \le \frac{1}{2\pi}\partial^2\Omega_{\overline{\rho}}^{Wh}(t) \le 3b_2$. □

B.3 *Proof of Proposition 2.7.* — The proof will follow from the following two lemmas.

Lemma **B.1.** — *Let $\beta \in \mathbf{R}$, $\nu > 0$; if for some $t + is \in U$ $(t, s \in \mathbf{R})$ one has*

$$|\omega(t + is) - \beta| < \nu,$$

then

$$\begin{cases} |\omega(t) - \beta| \le (7/6)\nu \\ |s| \le (4/3)A\nu. \end{cases} \tag{B.458}$$

Proof. — Since ω is holomorphic on U one has for any $z \in U$, $\overline{\partial}\omega(z) = 0$ (we use in this proof the usual notations $\overline{\partial} = (1/2)(\partial_t + i\partial_s)$ and $\partial = (1/2)(\partial_t - i\partial_s)$). For any point $z \in \mathbf{D}(0,\rho)$ one has (*cf.* Lemma 2.1)

$$\mathrm{dist}(z, U) \le 2\,\underline{a}(U)$$

and from the fact that $\|\mathrm{D}\overline{\partial}\omega\| \le \mathrm{B}$ we thus get using condition (2.60)

$$\|\overline{\partial}\omega\|_{C^0(\mathbf{D}(0,\rho))} \le \underline{a}(\mathrm{U})\mathrm{B} \le (8\mathrm{A})^{-1}. \tag{B.459}$$

Now, we write

$$\omega(t+is) - \beta = \omega(t) - \beta + \partial\omega(t)\cdot(is) + \overline{\partial}\omega(t)\cdot(-is) + \mathrm{O}(s^2) \tag{B.460}$$

where

$$|\mathrm{O}(s^2)| \le \|\mathrm{D}^2\omega\|_{C^0(\mathbf{D}(0,\rho))} \times s^2 \le \mathrm{B}\times\rho\times s \le (8\mathrm{A})^{-1}\times s \tag{B.461}$$

(*cf.* (2.60)). Note that since ω is real-symmetric, $\partial\omega(r)$ and $\overline{\partial}\omega(r)$ are real when r is real. Hence if $|\omega(t+is)-\beta| < \nu$ for some $t+is \in \mathrm{U}$, one gets by taking the imaginary part in (B.460), using (B.459), $\partial\omega(t) \in [\mathrm{A}^{-1}, \mathrm{A}]$ and (B.461), that

$$|s| \le (4/3)\mathrm{A}\nu. \tag{B.462}$$

This, (B.461) and taking the real part of (B.460) show that

$$|\omega(t)-\beta| \le |\Re(\omega(t+is)-\beta)| + |\Re(\mathrm{O}(s^2))| \le (7/6)\nu. \tag{B.463}$$ □

Because $t\mapsto\omega(t)$ is increasing with a derivative bounded below by A^{-1} (this is the twist condition) the set of $t\in]-\rho,\rho[:=\mathbf{D}(0,\rho)\cap\mathbf{R}$ such that $|\omega(t)-\beta|\le(7/6)\nu$ is a (possibly empty) interval I_β of length $\le (7/3)\mathrm{A}\nu$.

Lemma **B.2.** — *Let* $\nu\in]0,(6\mathrm{A}^2\mathrm{B})^{-1}[$. *If* I_β *is not empty there exists a unique* $c_\beta\in]-\rho-2\mathrm{A}\nu,\rho+2\mathrm{A}\nu[$ *such that*

$$\omega(c_\beta)=\beta, \qquad \omega(\overline{\mathbf{D}}(c_\beta,3\mathrm{A}\nu))\supset\overline{\mathbf{D}}(\beta,\nu).$$

Proof. — The uniqueness of c_β comes from the fact that $\mathbf{R}\ni t\mapsto\omega(t)\in\mathbf{R}$ is increasing. For the existence of c_β we just notice that if $\mathrm{I}_\beta\subset]-\rho,\rho[$ there is nothing to prove (notice that ω is increasing) and otherwise for some $\varepsilon\in\{-1,1\}$ $|\omega(\varepsilon\rho)-\beta|\le(7/6)\nu$. But then, the fact that $\mathrm{A}^{-1}\le\partial\omega(t)\le\mathrm{A}$ shows the existence of a unique $c_\beta\in[-\rho-(7/6)\mathrm{A}\nu,\rho+(7/6)\mathrm{A}\nu]$ such that $\omega(c_\beta)=\beta$.

If we set $a=\partial\omega(c_\beta)$ and $b=\overline{\partial}\omega(c_\beta)$ one has $a\ge\mathrm{A}^{-1}$, $|b|\le(7\mathrm{A})^{-1}$ (*cf.* (B.459) and the fact that $\nu\in]0,(6\mathrm{A}^2\mathrm{B})^{-1}[$) and the linear map $w\mapsto \mathrm{D}\omega(c_\beta)w = aw+b\overline{w}$ is invertible, the norm of its inverse being $\le(7/6)\mathrm{A}$. Next, we observe that because $|\omega(c_\beta+$

$w) - \omega(c_\beta) - D\omega(c_\beta)w| \leq B|w|^2/2$ and $\omega(c_\beta + w') - \omega(c_\beta + w) = \int_0^1 D\omega(c_\beta + w + t(w'-w))(w'-w)dt$, the map g

$$g : w \mapsto D\omega(c_\beta)^{-1}u - D\omega(c_\beta)^{-1}\Big(\omega(c_\beta + w) - \omega(c_b) - D\omega(c_\beta)w\Big)$$

is $(7/6)AB\Delta$-Lipschitz on $\{|w| \leq \Delta\}$ (some $\Delta > 0$) and sends $\{|w| \leq \Delta\}$ into itself provided $(7/6)AB\Delta \leq 1$ and $|u| \leq (6/14)A^{-1}\Delta$. In particular if one chooses $\Delta = 3A\nu$ the map g is $(1/2)$-contracting (remember $\nu \in]0, (6A^2B)^{-1}[$) and the Contraction Mapping Theorem shows that for any $|u| \leq \nu \leq (9/7)\nu$ there exists a unique $|w| \leq 3A\nu$ such that $\omega(c_\beta + w) = \beta + u$. □

We can now prove Proposition 2.7. We first observe that the computations done in the proof of Lemma B.2 show by the same token that the map $w \mapsto D\omega(0)^{-1}(\omega(w) - \omega(0) - D\omega(0)w)$ is $((7/6)AB\rho)$-Lipschitz on $\mathbf{D}(0, \rho)$ hence contracting from (2.60). This implies that ω is injective when restricted to $\mathbf{D}(0, \rho) \supset U$. Assume that (2.61) is satisfied for no $z \in U$. Then Lemma B.1 tells us that there exists $t + is \in U$ such that (B.458) holds and in particular that the interval I_β of Lemma B.2 is not empty. Applying this latter lemma and using the injectivity of ω when restricted to U we see that $\omega(U \setminus \overline{\mathbf{D}}(c_\beta, 3A\nu)) \subset \mathbf{C} \setminus \overline{\mathbf{D}}(\beta, \nu)$ which is the searched for conclusion. □

Appendix C: Illustration of the screening effect

We describe in this section an example where the screening effect mentioned in Section 3.2 is effective. Consider U as in (3.73)

$$\textbf{(C.464)} \qquad U = \mathbf{D}(0, \rho) \setminus \bigcup_{j=1}^{N} \overline{\mathbf{D}(c_j, \rho_j)} \supset \mathbf{D}(0, \sigma)$$

$(j = 1, \dots, N, \mathbf{D}(c_j, \rho_j) \subset \mathbf{D}(0, \rho), \rho_j \ll \sigma)$, with

$$\rho = 1, \qquad \sigma = e^{-N^\beta} < 1/10, \qquad \rho_j = \rho_1 := e^{-N^\gamma} \quad (j \in \{1, \dots, N\}),$$

$c_j = (1/4) + (2j-1)/(4N) \in [1/4, 3/4]$. We assume $N \gg 1$. The function $\varphi(\cdot) := \omega_{U \setminus \overline{\mathbf{D}(0,\sigma)}}(\cdot, \partial\mathbf{D}(0, \sigma))$ is harmonic on $U \setminus \overline{\mathbf{D}(0, \sigma)}$, equal to 1 on $\partial\mathbf{D}(0, \sigma)$ and equal to 0 on $\partial U = \partial\mathbf{D}(0, 1) \bigcup_{j=1}^{N} \mathbf{D}(c_j, \rho_1)$. Note that the minimum value of the Dirichlet integrals

$$\begin{cases} \int_{U \setminus \overline{\mathbf{D}(0,\sigma)}} |\nabla\psi(x + iy)|^2 dxdy \\ \psi \in H^1(U \setminus \overline{\mathbf{D}(0, \sigma)}), \quad \psi_{|\,\partial\mathbf{D}(0,\sigma)} = 1, \quad \psi_{|\,\partial U} = 0 \end{cases}$$

(H^1 denotes the usual Sobolev space) is achieved at φ, hence there exists a constant C independent of N such that

$$\text{(C.465)} \qquad \int_{U \setminus \overline{\mathbf{D}(0,\sigma)}} |\nabla\varphi(x+iy)|^2 dxdy \leq C;$$

(the constant C is for example the Dirichlet integral of any fixed C^1 function ψ equal to 1 on $\partial\mathbf{D}(0,\sigma)$ and to 0 on $U \setminus \mathbf{D}(0, 1/5)$).

We now use a result by Rauch and Taylor [40] that we adapt to the case of the complex plane: let $\mathcal{C}_H$ be the holed rectangle $U \cap ([1,4,3/4] + \sqrt{-1}[-H,H])$ with

$$H = C_1\delta \ln(\delta/\rho_1), \qquad \text{where } \delta = 3/(8N),$$

C_1 being some large constant. The set $\mathcal{C}_H$ can be covered by the N 1-holed rectangles $U \cap ([c_j - \delta, c_j + \delta] + \sqrt{-1}[-H,H])$, $j \in \{1,\dots,N\}$, each point of $\mathcal{C}_H$ belonging to no more than two of these holed rectangles. An adaptation of Inequality (4.1) of [40] to the case of domains in $\mathbf{R}^2$ ($\simeq \mathbf{C}$) asserts that in this situation there exists a constant $C_2 > 0$ (independent of φ, H, N) such that

$$\frac{\int_{\mathcal{C}_H} |\nabla\varphi(x+iy)|^2 dxdy}{\int_{\mathcal{C}_H} |\varphi(x+iy)|^2 dxdy} \geq \frac{C_2^{-1}}{H(H+\delta\ln(\delta/\rho_1)}$$

which in view of (C.465) and the choices made for H and δ implies

$$\frac{1}{H}\int_{\mathcal{C}_H} |\varphi(x+iy)|^2 dxdy \leq CC_2(C_1+1)\delta\ln(\delta/\rho_1) \lesssim N^{-(1-\gamma-)},$$

because $\rho_1 = \exp(-N^\gamma)$. In particular, for any $\mu > 0$ there exists a positive constant C_μ and a set $\mathcal{C}_{H,\mu}$ of relative measure $1-\mu$ in $\mathcal{C}_H$ such that for any $z \in \mathcal{C}_{H,\mu}$

$$|\omega_{U\setminus\overline{\mathbf{D}(0,\sigma)}}(z, \partial\mathbf{D}(0,\sigma))| = |\varphi(z)| \leq C_\mu N^{-(1-\gamma-\mu)},$$

an inequality which is in sharp contrast with (3.71), especially if $1-\gamma-\mu > \beta$. Getting a useful estimate like (3.72) by this technique is therefore doomed to fail if $\alpha < 1-\gamma-\mu$.

Appendix D: First integrals of integrable flows

This section is dedicated to the proof of the following Lemma, on first integrals of the integrable flow $\phi^t_{J\nabla r}$, that was used in the proof of Lemma 5.1.

Lemma **D.1.** — *Let* U *be a* σ*-symmetric open connected set of* $\mathbf{D}$ *and* $F \in \mathcal{O}_\sigma(W_{h,U})$ *such that*

$$\text{(D.466)} \qquad \forall\, t \in \mathbf{R},\ F \circ \phi^t_{J\nabla r} = F.$$

Then, there exists $\widetilde{F} \in \mathcal{O}_\sigma(U)$ *such that on* $W_{h,U}$ *one has*

$$F = \widetilde{F} \circ r.$$

Proof. — The Lemma is clear when we are in the (AA) case since the identity $\forall\, t \in \mathbf{R}$ $F(\theta + t, r) = F(\theta, r)$ clearly implies that F does not depend on θ. So we consider the (CC)-case.

We shall prove that for every $(z, w) \in W_{h,U}$ there exists an open neighborhood $V_{z,w}$ of (z, w) and a holomorphic function $f_{z,w}$ such that $F = f_{z,w} \circ r$ on $V_{z,w}$.

We consider three cases:

1) If $(z, w) = (0, 0) \in W_{h,U}$. One can write for μ small enough and $(z, w) \in \mathbf{D}(0, \mu)^2$, $F(z, w) = \sum_{k,l \in \mathbf{N}} F_{k,l} z^k w^l$. The identity (D.466) implies $\mathcal{M}_n(F) = 0$ for $n \neq 0$ hence from (5.103) one has $F(z, w) = \mathcal{M}_0(F)(z, w) = \sum_{k \in \mathbf{N}} F_{k,k}(zw)^k$ and we can choose $f_{0,0}(r) = \sum_{k \in \mathbf{N}} (ir)^k$.

2) If $zw = 0$, with for example $w \neq 0$. Then, from (5.106), $t \mapsto F(0, e^{it} w)$ is holomorphic with respect to $t \in \mathbf{R} + i] - \ln(e^h \rho^{1/2}/|w|), \infty[$ and constant on the real axis; it is hence constant on $\mathbf{R} + i] - \ln(e^h \rho^{1/2}/|w|), \infty[$. In particular taking $t = is$, $s \in \mathbf{R}_+$, gives $F(0, e^{-s} w) = F(0, w)$ and by making $s \to \infty$ we get $F(0, w) = F(0, 0)$ (notice that $(0, 0) \in W_{h,U}$ in that case). The same argument shows that for $(\tilde{z}, \tilde{w}) \in W_{h,U}$ the function $t \mapsto F(e^{-it}\tilde{z}, e^{it}\tilde{w})$ is constant on $t \in \mathbf{R} + i] - \ln(e^h \rho^{1/2}/|\tilde{w}|), \ln(e^h \rho^{1/2}/|\tilde{z}|)[$. Now if $(\tilde{z}, \tilde{w})$ is close enough to $(0, w)$, in particular if there exists $0 < s < \ln(e^h \rho^{1/2}/|\tilde{z}|)$ such that $|\tilde{w}|/\mu < e^s < \mu/|\tilde{z}|$, one has with $t = is$, $(e^{-it}\tilde{z}, e^{it}\tilde{w}) = (e^s \tilde{z}, e^{-s}\tilde{w}) \in \mathbf{D}(0, \mu)^2$. By (D.466) and point 1), one gets $F(\tilde{z}, \tilde{w}) = F(e^{-it}\tilde{z}, e^{it}\tilde{w}) = f_{0,0}(-i\tilde{z}\tilde{w})$.

3) Otherwise, we can assume that $zw \neq 0$. As before, we can argue that the function $t \mapsto g_{z,w}(t) := F(e^{-it} z, e^{it} w)$ is constant on the set

$$\mathbf{R} + i] - \ln(e^h \rho^{1/2}/|w|), \ln(e^h \rho^{1/2}/|z|)[.$$

Any point $(\tilde{z}, \tilde{w}) \in W_{h,U}$ which is close enough to (z, w) is of the form $\tilde{z} = e^{-it} z$, $\tilde{w} = e^{it} \lambda w$, t close to 0 and λ close to 1. We thus have

$$F(\tilde{z}, \tilde{w}) = F(e^{-it} z, \lambda e^{it} w) = F(z, \lambda w) = F(z, \tilde{z}\tilde{w} z^{-1}) = f_z(\tilde{z}\tilde{w})$$

where we have defined $f_z(r) = F(z, irz^{-1})$.

We have thus proven that for each $(z, w) \in W_{h,U}$ there exist a neighborhood $V_{z,w}$ and a holomorphic function $f_{z,w}$ such that $F = f_{z,w} \circ r$ on $V_{z,w}$. Now if $f_{z,w} \circ r = f_{z',w'} \circ r$ on a nonempty open set, the function $f_{z,w}$ and $f_{z',w'}$ coincide on a nonempty open set and thus there exists a holomorphic extension of $f_{z,w,z',w'}$ of these two functions such that $f_{z,w,z',w'} \circ r = f_{z,w} \circ r = f_{z',w'} \circ r$ on $V_{z,w} \cap V_{z',w'}$. We can now conclude by using the connectedness of U. □

Appendix E: (Formal) Birkhoff Normal Forms

Our aim in this Section is to recall the proof of the existence and uniqueness of the formal BNF, Propositions 6.1, 6.2. This is of course a standard topic but we tried to develop here a framework that is convenient for the proof of Lemma 6.3. We mainly concentrate on the (AA)-case since the formalism in the (CC)-case is very similar to the one developed by Pérez-Marco in [36].

E.1 *Formal preliminaries.*

E.1.1 *Formal series.* — Let $\mathbf{A}$ be a commutative ring and $\mathbf{A}[[X_1,\dots,X_{d'}]]$ ($d'\in\mathbf{N}^*$) the ring of formal power series $\sum_{n\in\mathbf{N}^{d'}} a_nX^n$, $a_n\in\mathbf{A}$, $X^n=X_1^{n_1}\cdots X_{d'}^{n_{d'}}$ (for short $X=(X_1,\dots,X_{d'})$). We denote by $v(A)=\min\{|n|=n_1+\cdots+n_{d'},\ a_n\neq 0\}$ the valuation of an element $A=\sum_{n\in\mathbf{N}^{d'}} a_nX^n$ and if $B=(B_1,\dots,B_{d'})\in(\mathbf{A}[[X]])^{d'}$ we define $v(B)=\min_l v(B_l)$.

For any $k\in\mathbf{N}$ we define $[A]_k=\sum_{|n|=k} a_kX^k$ the homogenous part of A of degree k and we set $[A]_{\le k}=\sum_{l=0}^{k}[A]_l$ (resp. $[A]_{\ge k}=\sum_{l=k}^{\infty}[A]_l$).

As usual the product of $A=\sum_{n\in\mathbf{N}^{d'}} a_nX^n$ and $B=\sum_{n\in\mathbf{N}^{d'}} b_nX^n$ is $AB=\sum_{n\in\mathbf{N}^{d'}}(\sum_{k+l=n} a_kb_l)X^n$ and the derivative $\partial_{X_l}A=\sum_{n\in\mathbf{N}^{d'}} n_la_nX^{n'}$ with $n'_j=n_j$ if $j\neq l$ and $n'_l=n_l-1$ is $n_l\ge 1$ (if $n_l=0$ the derivative of the corresponding monomial is zero). Note that if $A_i\in\mathbf{A}[[X]]$, $1\le i\le j$ one has

$$(\mathbf{E.467})\qquad [A_1\dots A_j]_k=\sum_{k_1+\cdots+k_j=k}[A_1]_{k_1}\cdots[A_j]_{k_j}.$$

When $A=\sum_{n\in\mathbf{N}^{d'}} a_nX^n\in\mathbf{A}[[X]]$ and $B\in(\mathbf{A}[[X]])^{d'}$, $v(B)\ge 1$ one can define

$$A\circ B=\sum_{n\in\mathbf{N}^{d'}} a_nB^n.$$

If moreover $\mathbf{A}$ is endowed with *derivations* $\delta_i:\mathbf{A}\to\mathbf{A}$, $1\le i\le d'$, (it means $\delta_i(a+b)=\delta_ia+\delta_ib$, $\delta_i(ab)=(\delta_ia)b+a(\delta_ib)$) we define (*cf.* Taylor formula) [42] for each $a\in\mathbf{A}$ and $B\in(\mathbf{A}[[X]])^{d'}$, $v(B)\ge 1$

$$(\mathbf{E.468})\qquad a\circ_\delta B=\sum_{k\in\mathbf{N}^{d'}}(1/k!)(\delta^ka)B^k\in\mathbf{A}[[X]].$$

Similarly, if $A=\sum_{k\in\mathbf{N}^{d'}} a_kX^k$, $B, C\in(\mathbf{A}[[X]])^{d'}$, $v(B)\ge 1$, $v(C)\ge 1$, we can also define

$$(\mathbf{E.469})\qquad A\circ_\delta(B,C)=\sum_{n\in\mathbf{N}^{d'}}(a_n\circ_\delta B)C^n.$$

[42] We use a multi-index notation, $k=(k_1,\dots,k_{d'})$, $B^k=B_1^{k_1}\cdots B_{d'}^{k_{d'}}$, $k!=k_1!\cdots k_{d'}!$, $\delta^k=\delta_1^{k_1}\cdots\delta_{d'}^{k_{d'}}$.

Lemma **E.1.** — *For $k \in \mathbf{N}^*$, $v(A) \geq 1$, $v(B) \geq 1$, $v(C) \geq 2$, $[A \circ_\delta (B, X + C) - A]_k$ is a polynomial in the coefficients of $[\delta^l A]_{k_1}, [B]_{k_2}, [C]_{k_3}$ for $k_1 + k_2 + k_3 \leq k - 1$, $|l| \leq k$ (this polynomial being with rational coefficients).*

Proof. — Since $(a_n \circ_\delta B)(X + C)^n - a_n X^n = (a_n \circ_\delta B)((X + C)^n - X^n) + X^n((a_n \circ_\delta B) - a_n)$

$$A \circ_\delta (B, X + C) - A = (I) + (II)$$

$$(I) := \sum_{\substack{|l|\geq 0 \\ |n|\geq 1 \\ m\leq n,\ |m|\geq 1}} \binom{n}{m} \frac{\delta^l a_n}{l!} B^l C^m X^{n-m}, \qquad (II) = \sum_{\substack{|l|\geq 1 \\ |n|\geq 1}} \frac{\delta^l a_n}{l!} B^l X^n$$

and one can conclude using (E.467). □

Assume now that $(\mathbf{A}, \delta)$ is endowed with a *translation* by which we mean an action τ of an abelian group (we suppose it is $(\mathbf{R}^d, +)$) on $\mathbf{A}$ that commutes with the derivations δ_i.

E.1.2 *Formal diffeomorphisms.* — A *formal diffeomorphism* of $\mathbf{A}[[X]]$ is a triple (α, A, B) (we denote it by $f_{\alpha,A,B}$) with $A, B \in (\mathbf{A}[[X]])^d$ with $v(B) \geq 2$ and where $v(A) \geq 1$ and $\alpha \in \mathbf{R}^d$. We can define the composition of two such objects:

$$f_{\varepsilon,E,D} = f_{\gamma,C,D} \circ f_{\alpha,A,B} \iff$$

$$\varepsilon = \alpha + \gamma, \qquad E = A + (\tau_{-\alpha} C) \circ_\delta (A, X + B),$$

$$F = B + (\tau_{-\alpha} D) \circ_\delta (A, X + B)$$

with $v(E) \geq 1$ and $v(F) \geq 2$. One can check that the usual algebraic rules for compositions are satisfied and that each such diffeomorphism has an inverse for composition.

Remark **E.1.** — One of the example we have in mind is the following. Take $d' = d \in \mathbf{N}^*$, $\mathbf{A} = C^\omega(\mathbf{T}^d)$ the ring of real analytic functions on $\mathbf{T}^d$ (taking real values on the real axis) and the ring of formal power series is $\mathbf{A}[[r]] = \{\sum_{n\in\mathbf{N}^d} a_n(\theta) r^n,\ a_n \in C^\omega(\mathbf{T}^d)\}$, $r = (r_1, \ldots, r_d)$. The derivations in this case are $\delta_i a = \partial_{\theta_i} a$ if $a : (\theta_1, \ldots, \theta_d) \to \mathbf{R}$ is in $C^\omega(\mathbf{T}^d)$, the translation is $\tau_\alpha a = a(\cdot - \alpha)$ $(\alpha \in \mathbf{R}^d)$ and the formal map $f_{\alpha,A,B}$ can be written under the more suggestive form

$$f_{(\alpha,A,B)}(\theta, r) = (\theta + \alpha + A(\theta, r), r + B(\theta, r))$$

as a formal diffeomorphism of $\mathbf{T}^d \times \mathbf{R}^d$.

E.1.3 *Degree.* — In case we can assign a *degree* $\deg(a)$ to each element a of the ring $\mathbf{A}$ (it satisfies by definition $\deg(0) = -\infty$, for all $a, b \in \mathbf{A}$, $\deg(a+b) = \max(\deg(a), \deg(b))$ and $\deg(ab) = \deg(a) + \deg(b)$) we can associate to each *weight* $p : \mathbf{N}^{d'} \to \mathbf{N}$ the set

$$\mathcal{C}(p) = \left\{ \sum_{n \in \mathbf{N}^{d'}} a_n X^n \in \mathbf{A}[[X]],\ \forall\, n \in \mathbf{N}^{d'},\ \deg(a_n) \le p(n) \right\}. \tag{E.470}$$

By extension if $B = (B_1, \dots, B_{d'}) \in (\mathbf{A}[[X]])^{d'}$ we say that B is in $\mathcal{C}(p)$ if each $B_l \in \mathcal{C}(p)$, $1 \le l \le d'$. If $p, q : \mathbf{N}^d \to \mathbf{N}$ we define

$$p * q(n) = \max_{(k,l) \in \mathbf{N}^{d'}, k+l=n} (p(k) + q(l)).$$

In particular if

$$\overline{p}(n) := |n| = n_1 + \cdots + n_{d'}, \qquad n = (n_1, \dots, n_{d'}) \in \mathbf{N}^{d'} \tag{E.471}$$

one has $\overline{p} * \overline{p} = \overline{p}$ and $(\overline{p} - 1)^{*m} = \overline{p} - m$.

We say that the degree deg is *compatible* with the derivations δ_i and the translation τ if for any $\alpha \in \mathbf{R}^d$, $1 \le i \le d'$, $\deg(\tau_\alpha \delta_i a) \le \deg(a)$.

Remark **E.2.** — The relevant example for our purpose (proof of Lemma 6.3) will be the following. Take $d' = d$, $\mathbf{A} = C^\omega(\mathbf{T}^d)[t]$ the set of polynomials in t with coefficients in $C^\omega(\mathbf{T}^d)$, $F^{(t)}(\theta) = a_0(\theta) + \cdots + a_n(\theta) t^n$, $a_j \in C^\omega(\mathbf{T}^d)$, $0 \le j \le n$, $n \in \mathbf{N}$ and $a_n \ne 0$. The derivations δ_i, $1 \le i \le d$ are defined by $\delta_i F^{(t)}(\theta) = (\partial_{\theta_i} a_0)(\theta) + \cdots + (\partial_{\theta_i} a_n)(\theta) t^n$, the translations $\tau_\alpha F^{(t)}(\theta) = a_0(\theta - \alpha) + \cdots + a_n(\theta - \alpha) t^n$ and the degree $\deg F^t = n$ is compatible with both of them.

The following facts are easily checked. Assume that $\mathbf{A}$ is a ring with derivations δ_i, $1 \le i \le d'$ and a compatible degree deg and let $(p_l)_{l \in \mathbf{N}}$ be weights.

(1) If $A, B \in \mathbf{A}[[X]]$, $A \in \mathcal{C}(p_1)$, $B \in \mathcal{C}(p_2)$ one has $AB \in \mathcal{C}(p_1 * p_2)$.
(2) If $A_l \in \mathbf{A}[[X]]$, $A_l \in \mathcal{C}(p_l)$, $\lim_{l \to \infty} v(A_l) = \infty$ one has $\sum_{l \in \mathbf{N}} A_l \in \mathcal{C}(\max_l p_l)$.

Let p be a weight such that $p * p \le p$. Using (E.468), points (1) and (2) we have

(3) If $a \in \mathbf{A}$, $B \in \mathbf{A}[[X]]$, $B \in \mathcal{C}(p)$ then one has $a \circ_\delta B - a \in \mathcal{C}(\deg(a) + p)$.

Lemma **E.2.** — *If* $A \in \mathbf{A}[[X]]$, $B, C \in (\mathbf{A}[[X]])^{d'}$, $v(B) \ge 1$, $v(C) \ge 1$ *with* $A \in \mathcal{C}(\overline{p} - c_A)$, $B \in \mathcal{C}(\overline{p} - c_B)$, $C \in \mathcal{C}(\overline{p} - 1)$, $\min(c_A, c_B) \ge 0$, *then* $A \circ_\delta (B, C) - A \circ C \in \mathcal{C}(\overline{p} - c_A - c_B)$ *and* $A \circ_\delta (B, C) \in \mathcal{C}(\overline{p} - c_A)$.

Proof. — Recall that $A \circ_\delta (B, C) = \sum_{n \in \mathbf{N}^{d'}} (a_n \circ_\delta B) C^n$. From point (3) $a_n \circ_\delta B - a_n \in \mathcal{C}(\deg(a_n) + \overline{p} - c_B)$ and from point (1) $(a_n \circ_\delta B) C^n - a_n C^n \in \mathcal{C}((\deg(a_n) + \overline{p} - c_B) * (\overline{p} - 1)^{*|n|}) \subset \mathcal{C}((\deg(a_n) + \overline{p} - c_B) * (\overline{p} - |n|))$ hence $(a_n \circ_\delta B) C^n - a_n C^n \in \mathcal{C}(\overline{p} - c_A - c_B)$. By using point (2) we have $A \circ_\delta (B, C) - A \circ C \in \mathcal{C}(\overline{p} - c_A - c_B)$. A similar argument shows that $A \circ C \in \mathcal{C}(\overline{p} - c_A)$ whence the conclusion. □

Before stating the next lemma we introduce the following definition: we say that a formal diffeomorphism $f_{\alpha,A,B}$ is in $\mathcal{D}(\overline{p} - 1)$ if $A \in \mathcal{C}(\overline{p}) \cap O(r)$, $B \in \mathcal{C}(\overline{p} - 1) \cap O^2(r)$.

Lemma **E.3.** — *One has*

(1) *Let* $H \in \mathcal{C}(\overline{p} - c)$, $c = 0, 1$, *and* $f_{\alpha,A,B} \in \mathcal{D}(\overline{p} - 1)$. *Then* $H \circ f_{\alpha,A,B} \in \mathcal{C}(\overline{p} - c)$.

(2) *The composition of two formal diffeomorphisms in* $\mathcal{D}(\overline{p} - 1)$ *is in* $\mathcal{D}(\overline{p} - 1)$.

(3) *The inverse for the composition of a diffeomorphism of* $\mathcal{D}(\overline{p} - 1)$ *is in* $\mathcal{D}(\overline{p} - 1)$.

(4) *If* $f_{\alpha,A,B}^{-1} = f_{\tilde{\alpha},\tilde{A},\tilde{B}}$, *then for any* $k \geq 1$, $[\tilde{A}]_k$, $[\tilde{B}]_k$, *are polynomials in the coefficients of* $[\tau_{m_1\alpha}\delta^{l_1}A]_{k_1}$, $[\tau_{m_2\alpha}\delta^{l_2}B]_{k_2}$, $k_1, k_2 \leq k$, $l_1, l_2 \leq k$, $|m_1|, |m_2| \leq k$.

Proof. — Items 1 and 2 are consequences of Lemma E.2.

For point 3 we just have to prove the result when $\alpha = 0$. Let us denote by $\mathcal{U}$ the operator $H \mapsto H \circ f_{0,A,B}$. Note that $v((\mathcal{U} - id)H) \geq v(H) + 1$ hence the series $\tilde{H} := \sum_{k=0}^{\infty}(\mathcal{U} - id)^k H$ converges in $\mathbf{A}[[r]]$ and from 1 and 2 one sees that if $H \in \mathcal{C}(\overline{p} - c)$, $c = 0, 1$, the same is true for $\tilde{H}$. To conclude we observe that $-(f_{0,A,B}^{-1} - id) = (\mathcal{U} - id)(f_{0,A,B}^{-1} - id) + (f_{0,A,B} - id)$ hence

$$-(f_{0,A,B}^{-1} - id) = \sum_{k=0}^{\infty}(\mathcal{U} - id)^k(f_{0,A,B} - id). \tag{E.472}$$

Finally point 4 is a consequence of (E.472) and Lemma E.1. □

E.2 *Formal Birkhoff Normal Forms.* — From now on we work in the setting of Remark E.2.[43]

E.2.1 *Formal exact symplectic diffeomorphism.* — If $F \in \mathbf{A}[[r]]$, $F = \langle \alpha, r \rangle + O^2(r)$ with $\alpha \in \mathbf{R}^d$ we define the formal diffeomorphism $f_F(\theta, r) = (\theta + A(\theta, r), r + B(\theta, r))$ as suggested by the implicit relation

$$\varphi = \theta + \partial_R F(\theta, R), \qquad r = R + \partial_\theta F(\theta, R), \qquad f_F(\theta, r) = (\varphi, R)$$

or more formally

$$A(\theta, r) = \partial_r F(\theta, r + B(\theta, r)), \qquad 0 = B(\theta, r) + \partial_\theta F(\theta, r + B(\theta, r))$$

$$A = \partial_r F \circ_\delta (0, B), \qquad 0 = B + \partial_\theta F \circ_\delta (0, B) \qquad \text{(cf. E.469).)}$$

[43] Note that to prove the existence and uniqueness of the formal BNF of Section E.2.2 it would be enough to work with $\mathbf{A} = \mathbf{C}^\omega(\mathbf{T}^d)$.

In this situation we use the more intuitive notations $R(\theta, r) = r + B(\theta, r)$, $\varphi(\theta, r) - \theta = A(\theta, r)$. We shall call such formal diffeomorphisms f_F *formal exact symplectic diffeomorphisms*. The set of all such diffeomorphisms is a group under composition. Let us define

$$\mathcal{E}(\bar{p}-1) := \{f_F,\ F = \langle \alpha, r\rangle + \sum_{|n|\geq 2} F_n(\theta) r^n \in \mathbf{A}[[r]],\ F \in \mathcal{C}(\bar{p}-1)\}.$$

The following result is then a consequence of Lemma E.3.

Lemma **E.4.** — *The set* $\mathcal{E}(\bar{p}-1)$ *is a subset of* $\mathcal{D}(\bar{p}-1)$ *stable by composition and inversion.*

Remark **E.3.** — In the (CC)-case the relevant choice for $\mathbf{A}$ is $\mathbf{C}[[t]]$ and the set of formal series is $\mathbf{A}[[z, w]]$. One can extend in this context the notion of σ_2-symmetry. If $F = \sum_{(n,m)\in \mathbf{N}^d\times\mathbf{N}^d} F_{n,m} z^m w^m \in \mathbf{A}[[z, w]]$ we say it is σ_2-symmetric (recall $\sigma_2(z, w) = (i\bar{w}, i\bar{z})$) if $\overline{F}_{n,m} = (i)^{|n|+|m|} F_{m,n}$ for all $(n, m) \in \mathbf{N}^d \times \mathbf{N}^d$ ($\overline{F}_{n,m}$ is the complex conjugate of $F_{n,m}$ and $i = \sqrt{-1}$). Similarly one can define the notion of σ_2-symmetric formal diffeomorphism. If F is σ_2-symmetric then f_F is σ_2-symmetric.

E.2.2 *Existence and uniqueness of the formal BNF.* — We prove at the end of this subsection that given $f_{2\pi\langle\omega_0, r\rangle+F}$ we can find $B(r) = 2\pi\langle\omega_0, r\rangle + O^2(r) \in \mathbf{R}[[r]]$ and a formal exact symplectic diffeomorphism $f_Z = id + O^2(r)$, $Z = O^2(r)$ which is *normalized* in the sense that

$$\int_{\mathbf{T}^d} Z(\varphi, Q)\, d\varphi = 0, \tag{E.473}$$

such that

$$f_Z \circ f_{2\pi\langle\omega_0, r\rangle+F}(\theta, r) = f_B \circ f_Z(\theta, r). \tag{E.474}$$

Moreover, Z and B are uniquely determined by F.

We use the notations $f_{2\pi\langle\omega_0, r\rangle+F} : (\theta, r) \mapsto (\varphi, R)$, $f_Z : (\varphi, R) \mapsto (\psi, Q)$ so that

$$\psi = \varphi + \partial_Q Z(\varphi, Q), \qquad R = Q + \partial_\varphi Z(\varphi, Q)$$
$$\varphi = \theta + 2\pi\omega_0 + \partial_R F(\theta, R), \qquad r = R + \partial_\theta F(\theta, R).$$

Using the relation $R = Q + \partial_\varphi Z(\theta + 2\pi\omega_0 + \partial_R F(\theta, R), Q)$ and the fact that $g : (\theta, Q, R) \mapsto (\theta, Q, R - Q - \partial_\varphi Z(\theta + 2\pi\omega_0 + \partial_R F(\theta, R), Q))$ is a formal diffeomorphism[44] we can define $R(\theta, Q) = Q + O^2(Q)$ by $(\theta, Q, R(\theta, Q)) = g^{-1}(\theta, Q, 0)$.

[44] Defined on $\mathbf{A}[[Q, R]]$.

Lemma **E.5.**

(1) For any $k \geq 1$, *the coefficients of* $[\mathrm{R}(\theta, \mathrm{Q})]_k$ *are polynomials in the coefficients of* $[\tau_{2\pi m_1\omega_0}\delta^{l_1}\mathrm{F}]_{k_1}$, $[\tau_{2\pi m_2\omega_0}\delta^{l_2}\mathrm{Z}]_{k_2}$, $k_1, k_2, l_1, l_2, |m_1|, |m_2| \leq k$.

(2) If $\mathrm{F}, \mathrm{Z} \in \mathcal{C}(\overline{p}-1)$, *the formal diffeomorphism* $(\theta, \mathrm{Q}) \mapsto (\theta, \mathrm{R}(\theta, \mathrm{Q}))$ *is in* $\mathcal{D}(\overline{p}-1)$.

Proof. — These are consequences of Lemma E.3. □

Let $f_{\mathrm{Z}}(\theta, r) = (\theta', r')$; from the formal conjugation relation (E.474) we get $f_{\mathrm{B}} \circ f_{\mathrm{Z}}(\theta, r) = (\theta' + \nabla\mathrm{B}(r'), r') = (\psi, \mathrm{Q})$ hence $\mathrm{Q} = r'$ and $\theta' = \theta + \partial_{\mathrm{Q}}\mathrm{Z}(\theta, \mathrm{Q})$. We thus have

$$\theta + \partial_{\mathrm{Q}}\mathrm{Z}(\theta, \mathrm{Q}) + \nabla\mathrm{B}(\mathrm{Q}) = \varphi + \partial_{\mathrm{Q}}\mathrm{Z}(\varphi, \mathrm{Q})$$

and using the relations between φ, θ yields

$$-\partial_{\mathrm{R}}\mathrm{F}(\theta, \mathrm{R}) = \partial_{\mathrm{Q}}\mathrm{Z}\Big(\theta + 2\pi\omega_0 + \partial_{\mathrm{R}}\mathrm{F}(\theta, \mathrm{R}), \mathrm{Q}\Big) - \partial_{\mathrm{Q}}\mathrm{Z}(\theta, \mathrm{Q}) \\ - (\nabla\mathrm{B}(\mathrm{Q}) - 2\pi\omega_0)$$

that we can write

$$-\partial_{\mathrm{Q}}\mathcal{F}(\mathrm{F}, \mathrm{Z}) = \partial_{\mathrm{Q}}\mathrm{Z}(\theta + 2\pi\omega_0, \mathrm{Q}) - \partial_{\mathrm{Q}}\mathrm{Z}(\theta, \mathrm{Q}) - (\partial_{\mathrm{Q}}\mathrm{B}(\mathrm{Q}) - 2\pi\omega_0) \tag{E.475}$$

where $\mathcal{F}(\mathrm{F}, \mathrm{Z}) = \mathrm{O}^2(r)$ is uniquely defined (note that the RHS of (E.475) is $\mathrm{O}(r)$) by

$$\partial_{\mathrm{Q}}\mathcal{F}(\mathrm{F}, \mathrm{Z}) = \partial_{\mathrm{Q}}\mathrm{F}(\theta, \mathrm{R}(\theta, \mathrm{Q})) \\ + \Big(\partial_{\mathrm{Q}}\mathrm{Z}\Big(\theta + 2\pi\omega_0 + \partial_{\mathrm{Q}}\mathrm{F}(\theta, \mathrm{R}(\theta, \mathrm{Q})), \mathrm{Q}\Big) - \partial_{\mathrm{Q}}\mathrm{Z}(\theta + 2\pi\omega_0, \mathrm{Q})\Big). \tag{E.476}$$

We thus have

$$-\mathcal{F}(\mathrm{F}, \mathrm{Z}) = \mathrm{Z}(\theta + 2\pi\omega_0, \mathrm{Q}) - \mathrm{Z}(\theta, \mathrm{Q}) - (\mathrm{B}(\mathrm{Q}) - 2\langle\pi\omega_0, \mathrm{Q}\rangle). \tag{E.477}$$

Lemma **E.6.**

(1) For any $k \geq 1$, *the coefficients of* $[\mathcal{F}(\mathrm{F}, \mathrm{Z}) - \mathrm{F}]_k$ *are polynomials in the coefficients of* $[\tau_{2\pi m_1\omega_0}\delta^{l_1}\mathrm{F}]_{k_1}$, $[\tau_{2\pi m_2\omega_0}\delta^{l_2}\mathrm{Z}]_{k_2}$, $k_1, k_2 \leq k-1$, $l_1, l_2 \leq k$, $|m_1|, |m_2| \leq k$.

(2) If $\mathrm{F}, \mathrm{Z} \in \mathcal{C}(\overline{p}-1)$, *one has* $\mathcal{F}(\mathrm{F}, \mathrm{Z}) \in \mathcal{C}(\overline{p}-1)$.

Proof. — This is a consequence of (E.476), Lemma E.5 and Lemma E.3. □

From (E.477) one thus has

$$\begin{cases} k = 2, & -[\mathrm{F}]_2(\theta, \mathrm{Q}) = [\mathrm{Z}]_2(\theta + 2\pi\omega_0, \mathrm{Q}) - [\mathrm{Z}]_2(\theta, \mathrm{Q}) - [\mathrm{B}]_2(\mathrm{Q}), \\ \forall\, k \geq 3 & -[\mathcal{F}(\mathrm{F}, \mathrm{Z})]_k(\theta, \mathrm{Q}) = [\mathrm{Z}]_k(\theta + 2\pi\omega_0, \mathrm{Q}) - [\mathrm{Z}]_k(\theta, \mathrm{Q}) - [\mathrm{B}]_k(\mathrm{Q}). \end{cases} \tag{E.478}$$

Before completing the proof of the existence and uniqueness of the Birkhoff Normal Form (E.474) we state the following result (the first part of which at least is classical; see for example [13]):

Lemma **E.7.** — *If* ω_0 *is Diophantine, for any* $G \in \mathbf{A}[[r]]$ *there exists a unique pair* (Z, B) *with* $Z \in \mathbf{A}[[r]]$ *normalized in the sense of (E.473) and* $B = B(r) \in \mathbf{R}[t][[r]]$ *such that*

$$G(\theta, Q) = Z(\theta + 2\pi\omega_0, Q) - Z(\theta, Q) + B(Q). \tag{E.479}$$

Furthermore: (1) $B(Q) = \int_{\mathbf{T}^d} G(\theta, Q)d\theta$ *and if* $G = [G]_k$ *one has* $Z = [Z]_k$, $B = [B]_k$ *and the coefficients of* $[Z]_k$ *are* $\mathbf{R}$*-linear functions of the coefficients of* $[G]_k$*; (2) if* $G \in \mathcal{C}(\bar{p} - 1)$ *then* $Z, B \in \mathcal{C}(\bar{p} - 1)$.

Proof. — If we denote by $\widehat{G}(l, Q) = (2\pi)^{-d} \int_{\mathbf{T}^d} G(\theta, Q)e^{-i\langle l,\theta\rangle}d\theta$ the l-th Fourier coefficient of $\theta \mapsto G(\theta, Q)$ ($l \in \mathbf{Z}^d$) then (E.479) follows if $B(Q) = \int_{\mathbf{T}^d} G(\theta, Q)d\theta$ and if $Z(\theta, Q)$ is defined by

$$Z(\theta, Q) = \sum_{l\in\mathbf{Z}^d \setminus \{0\}} (e^{2\pi i\langle l,\omega_0\rangle} - 1)^{-1}\widehat{G}(l, Q)e^{i\langle l,\theta\rangle}.$$

The conclusions of the Lemma are clear from the preceding expression. □

Proof of the existence and uniqueness of the BNF (E.474).

– *Uniqueness*: Equation (E.478), Lemma E.6 and Lemma E.7 show inductively that $[Z]_k$, $[B]_k$ are uniquely defined by $[F]_j$, $2 \le j \le k-1$. Hence, Z and B are unique.

– *Existence*: Define $[Z]_2$, $[B]_2$ by (E.478) and then inductively for $k \ge 3$, $[Z]_k$, $[B]_k$, by

$$-[\mathcal{F}(F, [Z]_{\le k-1})]_k(\theta, Q) = [Z]_k(\theta + 2\pi\omega_0, Q) - [Z]_k(\theta, Q) - [B]_k(Q) \tag{E.480}$$

where $Z_{\le k-1} = \sum_{l=2}^{k-1}[Z]_l$. Setting $F = \sum_{l=2}^{\infty}[Z]_l$, $B = \sum_{l=2}^{\infty}[B]_l$ one can check that (E.477) holds modulo $O^k(r)$ for any k and hence in $\mathbf{A}[[r]]$. □

E.3 *Proof of Lemma 6.3.* — We define $F^{(t)}(\theta, r) = tF(\theta, r) + (1-t)G(\theta, r)$ which is in $\mathbf{A}[[r]] \cap \mathcal{C}(\bar{p} - 1)$, $\mathbf{A} = C^{\omega}(\mathbf{T}^d)[t]$. Note that for any $k \ge 2$, $[F^{(t)}]_k \in \mathcal{C}(\bar{p} - 1)$. In particular, as a consequence of Lemmata E.6, item 2 and E.7, point (2), the sequences $[Z^{(t)}]_k$, $[B^{(t)}]_k$, inductively constructed in (E.480), are in $\mathcal{C}(\bar{p} - 1)$. Hence $B^{(t)}(r) := \sum_{n\in\mathbf{N}^d} b_n(t)r^n$ is in $\mathcal{C}(\bar{p} - 1)$ which by definition (*cf.* (E.470), (E.471)) means that the degree in t of each $b_n(t)$ is $\le |n| - 1$. □

Appendix F: Approximate Birkhoff Normal Forms

We give in this section the proofs of Propositions 6.4 and 6.5.

F.1 *A useful lemma.* — Let be given for each $\alpha \in]0, 1/2[$, a function $P_\alpha : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, $P_\alpha : (k, t) \mapsto P_\alpha(k, t)$ nondecreasing in each variable and assume that $(\varepsilon_{\alpha,k})_{k\in I_\alpha}$, $I_\alpha \subset \mathbf{N}$ is a sequence of nonnegative real numbers depending on $\alpha \in]0, 1/2[$ and defined inductively as long as a condition of the form

$$P_\alpha(k, \varepsilon_{\alpha,k}) < 1 \tag{F.481}$$

is satisfied (we assume that $\varepsilon_{\alpha,0}$ satisfies (F.481)). Let us call $J_\alpha = [\![0, k_\alpha^*]\!]$, $k_\alpha^* \geq 1$ the maximal set of integers $k \in \mathbf{N}$ for which $\varepsilon_{\alpha,k}$ is defined: this means that if $k \in J_\alpha$ and $\varepsilon_{\alpha,k}$ satisfies (F.481) then $k+1 \in J_\alpha$ (in particular $P_\alpha(k_\alpha^*, \varepsilon_{\alpha,k_\alpha^*}) \geq 1$). Let $\theta > 0$, $a > 0$ and $\bar{k}_{\theta,a} \in \mathbf{N}^*$ be such that

$$\forall k \in [\![0, \min(k_\alpha^*, \bar{k}_{\theta,a}) - 1]\!], \quad \varepsilon_{\alpha,k+1} \leq C_{\theta,a}\alpha^\theta \times \left(1 + \alpha^{-a}\sum_{j=0}^{k} \varepsilon_{\alpha,j}\right)\varepsilon_{\alpha,k}. \tag{F.482}$$

We have the following type of Gronwall Lemma:

Lemma **F.1.** — *Assume that*

$$(2C_{\theta,a}\alpha)^\theta < 1/2, \qquad \varepsilon_{\alpha,0} \leq \alpha^a/2, \qquad P_\alpha(\bar{k}_{\theta,a} + 1, \varepsilon_{\alpha,0}) < 1. \tag{F.483}$$

Then,

$$k_\alpha^* \geq \bar{k}_{\theta,a} \tag{F.484}$$

and

$$\forall\, k \in [0, \bar{k}_{\theta,a}] \cap \mathbf{N}, \quad \varepsilon_{\alpha,k} \leq (2C_{\theta,a}\alpha)^{\theta k}\varepsilon_{\alpha,0}. \tag{F.485}$$

Proof. — 1) Let $k_{\alpha,\theta,a}^* = \min(k_\alpha^*, \bar{k}_{\theta,a})$. We first prove that the set

$$K_{\alpha,\theta,a} = \{k \in [\![0, k_{\alpha,\theta,a}^*]\!],\ \varepsilon_{\alpha,k} > (2C_{\theta,a}\alpha)^{\theta k}\varepsilon_{\alpha,0}\}$$

is empty. If this were not the case we could define $k_{\alpha,\theta,a} = \inf K_{\alpha,\theta,a}$ and write

$$\forall\, k \in [\![0, k_{\alpha,\theta,a} - 1]\!], \quad \varepsilon_{\alpha,k} \leq (2C_{\theta,a}\alpha)^{\theta k}\varepsilon_{\alpha,0} \tag{F.486}$$

hence

$$\sum_{j=0}^{k} \varepsilon_{\alpha,j} \leq \frac{\varepsilon_{\alpha,0}}{1 - (2C_{\alpha,\theta,a}\alpha)^\theta} \leq 2\varepsilon_{\alpha,0}$$

and thus from (F.482) and (F.483), for all $0 \leq k \leq k_{\alpha,\theta,a} - 1$,

$$\varepsilon_{\alpha,k+1} \leq C_{\theta,a} \times \alpha^\theta(1 + 2\alpha^{-a}\varepsilon_{\alpha,0})\varepsilon_{a,k} \leq (2C_{\theta,a}\alpha^\theta)\varepsilon_{\alpha,k}.$$

This implies that for all $0 \leq k \leq k_{\alpha,\theta,a}$ one has

$$\varepsilon_{\alpha,k} \leq (2C_{\theta,a}\alpha)^{\theta k}\varepsilon_{\alpha,0}$$

and in particular $\varepsilon_{\alpha,k_{\alpha,\theta,a}} \leq (2C_{\theta,a}\alpha)^{\theta k_{\alpha,\theta,a}}\varepsilon_{\alpha,0}$. This contradicts the definition of $k_{\alpha,\theta,a}$ as $\inf K_{\alpha,\theta,a}$.

2) Since $K_{\alpha,\theta,a}$ is empty, one has

$$\text{(F.487)} \qquad \forall\, k \in [\![0, k^*_{\alpha,\theta,a}]\!], \quad \varepsilon_{\alpha,k} \leq (2C_{\theta,a}\alpha)^{k\theta}\varepsilon_{\alpha,0} \leq \varepsilon_{\alpha,0}.$$

But since $k^*_{\alpha,\theta,a} + 1 \leq \overline{k}_{\theta,a} + 1$ and $(2C_{\theta,a}\alpha) \leq 1$

$$\begin{aligned} P_\alpha(k^*_{\alpha,\theta,a} + 1, \varepsilon_{\alpha,k^*_{\alpha,\theta,a}}) &\leq P_\alpha(\overline{k}_{\theta,a} + 1, \varepsilon_{a,0}) \\ &< 1 \end{aligned}$$

This implies that $k^*_{\alpha,\theta,a} + 1 \in J_\alpha$ hence $k^*_{\alpha,\theta,a} < k^*_\alpha$ and from the definition of $k^*_{\alpha,\theta,a}$, we get $k^*_\alpha \geq \overline{k}_{\theta,a}$ which in its turn implies $k^*_{\alpha,\theta,a} = \overline{k}_{\theta,a}$. We have thus proven that for all $k \leq \overline{k}_{\theta,a}$ one has

$$\varepsilon_{\alpha,k} \leq (2C_{\theta,a}\alpha)^{k\theta}\varepsilon_{\alpha,0}. \qquad \square$$

F.2 *Proof of Proposition 6.4 (BNF, (CC)-Case).* — For n large enough we define

$$\text{(F.488)} \qquad \rho_0 = \frac{1}{q_n^6}, \qquad W_0 = e^h W_{h,\mathbf{D}(0,\rho_0)}$$

and for $k \geq 0$ we introduce the sequences[45]

$$\text{(F.489)} \qquad \delta_k = C^{-1}\frac{h}{(k+1)(\ln(k+2))^2}, \qquad \sum_{l=0}^{\infty}\delta_l \leq h/10$$

$$\text{(F.490)} \qquad \rho_k = \exp\Big(-\sum_{l=0}^{k-1}\delta_l\Big)\rho_0, \quad W_k = \exp\Big(-\sum_{l=0}^{k-1}\delta_l\Big)W_0.$$

Recall that $\overline{a}_1 = \max(2\overline{a} + 1, 30)$ and that for some $m \geq \overline{a}_1$ (*cf.* (6.141))

$$\text{(F.491)} \qquad F_0(z, w) = O^{2m}(z, w), \qquad \varepsilon_0 := \|F_0\|_{W_0} \lesssim \rho_0^m.$$

[45] The choice of these sequences, in particular the choice of a summable sequence δ_k, is not necessary (we only perform a finite number of conjugation steps) and is indeed not the best one insofar as it leads to a(n) (arbitrary small) loss β in the exponent. The "optimal" one is (after loosing a fixed fraction of h in the first step) to choose uniformly at each step $\delta_k \sim h/k^*$ so that $\sum_{0\leq k\leq k^*}\delta_k \sim h$ where k^* is the finite number of conjugation steps we perform.

We shall construct inductively for $k \geq 0$, sequences $Z_k \in \mathcal{O}_\sigma(W_k)$, $F_k \in \mathcal{O}_\sigma(W_k) \cap O^{k+2m}(z, w)$, $\Omega_k \in \mathcal{O}_\sigma(\mathbf{D}(0, \rho_k))$, $\Omega_k(r) = 2\pi\omega_0 r + O^2(r)$, such that

$$\Omega_0 = \Omega, \qquad F_0 = F$$

and for $k \geq 1$,

$$g_k^{-1} \circ \Phi_{\Omega_0} \circ f_{F_0} \circ g_k = \Phi_{\Omega_k} \circ f_{F_k}. \tag{F.492}$$

To do this we proceed the following way: assuming (F.492) holds and $\delta_k \leq q_n^{-1}$, we apply Proposition 5.3 with $\tau = 0$, $K/2 = N = q_n$ *cf.* (6.143), and define $Y_k \in \mathcal{O}_\sigma(W_k) \cap O^{k+2m}(z, w)$ (see Remark 5.3) satisfying

$$-[Q_0] \cdot Y_k = T_{q_n} F_k - \mathcal{M}_0(F_k), \qquad \|Y_k\|_{e^{-\delta_k/2}W_k} \lesssim q_n^2 \|F_k\|_{W_k} \tag{F.493}$$

where we denote

$$Q_0(r) = 2\pi\omega_0 r. \tag{F.494}$$

Using Lemma 5.4 we get (*cf.* formula (5.124))

$$\begin{aligned} f_{Y_k} \circ \Phi_{\Omega_k} \circ f_{F_k} \circ f_{Y_k}^{-1} &= \Phi_{\Omega_k + M(F_k)} \circ f_{F_k - \mathcal{M}_0(F_k) + [\Omega_k + M(F_k)] \cdot Y_k + \dot{\mathfrak{O}}_2^{(\overline{a})}(Y_k, F_k)} \\ &= \Phi_{\Omega_k + M(F_k)} \circ f_{R_{q_n} F_k + [\Omega_k + M(F_k)] \cdot Y_k - [Q_0] \cdot Y_k + \dot{\mathfrak{O}}_2^{(\overline{a})}(Y_k, F_k)}. \end{aligned}$$

Hence $f_{Y_k} \circ \Phi_{\Omega_k} \circ f_{F_k} \circ f_{Y_k}^{-1} = \Phi_{\Omega_{k+1}} \circ f_{F_{k+1}}$ with

$$\Omega_{k+1} - \Omega_k = M(F_k) \tag{F.495}$$

and, using the fact that $[\Omega_k + M(F_k)] \cdot Y_k - [Q_0] \cdot Y_k = O(|\nabla Y_k| \times |\partial(\Omega_k - Q_0) \circ r|) + \dot{\mathfrak{O}}_2^{(2)}(Y_k, F_k)$,

$$F_{k+1} = R_{q_n} F_k + \dot{\mathfrak{O}}_2^{(\overline{a})}(Y_k, F_k) + O\Big(|\nabla Y_k| \times |\partial(\Omega_k - Q_0) \circ r|\Big). \tag{F.496}$$

Notice that from (F.495) and the fact that for $0 \leq j \leq k-1$, $\mathcal{M}_0(F_j) = O^{j+2m}(z, w)$ (*cf.* the remark at the end of Section 5.1.1), hence $M(F)(r) = O^m(r)$; one thus has

$$\forall\, 0 \leq j \leq k, \qquad \Omega_k(r) - Q_0(r) = O(r^2). \tag{F.497}$$

Since $F_k, Y_k \in O^{k+2m}(z, w)$ we have (see Remarks 2.1, 5.3) $\dot{\mathfrak{O}}_2^{(\overline{a})}(Y_k, F_k) = O^{2k+4m-2\overline{a}}(z, w)$; also, $O(|\nabla Y| \times |\partial(\Omega_k - Q_0) \circ r|) = O^{k+2m+1}(z, w)$ and from (5.115) $R_{q_n} F_k = O^{q_n}(z, w)$ (if $q_n \geq m$). As a consequence, since $2k + 4m - 2\overline{a} \geq k + 1 + 2m$ ($m \geq 2\overline{a} + 1$) we see that

$F_{k+1} = O^{k+1+2m}(z, w)$. Furthermore since $\Omega_0(r) - Q_0(r) = O(r^2)$

$$\begin{aligned}\|\partial(\Omega_k - Q_0)\circ r\|_{W_k} &\leq \|\partial(\Omega_k - \Omega_0)\circ r\|_{W_k} + \|\partial(\Omega_0 - Q_0)\circ r\|_{W_k} \\ &\lesssim \|\partial(\Omega_k - \Omega_0)\circ r\|_{W_k} + \sup|r(W_k)| \\ &\lesssim \|\partial(\Omega_k - \Omega_0)\circ r\|_{W_k} + \rho_k\end{aligned}$$

and using (2.55)

$$\begin{aligned}\|\nabla Y_k\|_{e^{-\delta_k}W_k} &\lesssim \delta_k^{-1}\rho_k^{-1/2}\|Y_k\|_{e^{-\delta_k/2}W_k} \\ &\lesssim q_n^2\rho_k^{-1/2}\delta_k^{-1}\|F_k\|_{W_k}\end{aligned}$$

hence

$$|\nabla Y_k| \times |\partial(\Omega_k - Q_0| \lesssim q_n^2\rho_k^{-2}\delta_k^{-2}\|\Omega_k - \Omega_0\|_{\mathbf{D}(0,\rho_k)}\|F_k\|_{W_k} + q_n^2\delta_k^{-1}\rho_k^{1/2}\|F_k\|_{W_k}. \tag{F.498}$$

From (F.496), (F.493), (F.498), and (5.112) we get that, provided

$$\rho_k^{-\overline{a}}\delta_k^{-\overline{a}}q_n^2\|F_k\|_{W_k} < 1,$$

one has the inequalities

$$\begin{aligned}\|F_{k+1}\|_{e^{-\delta_k}W_k} \lesssim\ & \delta_k^{-1}e^{-q_n\delta_k/2}\|F_k\|_{W_k} + \rho_k^{-\overline{a}}\delta_k^{-\overline{a}}q_n^2\|F_k\|_{W_k}^2 \\ &+ q_n^2\rho_k^{-2}\delta_k^{-2}\|\Omega_k - \Omega_0\|_{\mathbf{D}(0,\rho_k)}\|F_k\|_{W_k} + q_n^2\delta_k^{-1}\rho_k^{1/2}\|F_k\|_{W_k}\end{aligned} \tag{F.499}$$

and

$$\|\Omega_{k+1} - \Omega_0\|_{\mathbf{D}(0,e^{-\delta_k}\rho_k)} \lesssim \sum_{j=1}^{k}\|F_j\|_{W_j}. \tag{F.500}$$

Let us define

$$s_k = \|\Omega_k - \Omega_0\|_{\mathbf{D}(0,\rho_k)}, \qquad \varepsilon_k = \|F_k\|_{W_k};$$

then, one has

$$\begin{cases}\varepsilon_{k+1} \lesssim \left(\delta_k^{-1}e^{-q_n\delta_k/2} + (\rho_k\delta_k)^{-\overline{a}}q_n^2\varepsilon_k + q_n^2(\rho_k\delta_k)^{-2}s_k + q_n^2\delta_k^{-1}\rho_k^{1/2}\right)\varepsilon_k \\ s_{k+1} \lesssim \sum_{j=0}^{k}\varepsilon_j\end{cases} \tag{F.501}$$

as long as

$$\rho_k^{-\overline{a}}\delta_k^{-\overline{a}}q_n^2\varepsilon_k < 1.$$

Let k^* be the largest integer for which the preceding sequences are defined and satisfy the stronger condition

$$\text{(F.502)} \qquad \forall\, k < k^*, \quad \mathrm{P}_{q_n}(k,\varepsilon_k) := \rho_0^{-1/5}\rho_k^{-\overline{a}}\delta_k^{-\overline{a}}q_n^2\varepsilon_k < 1.$$

From (F.489) for any $\sigma > 0$ one has $\delta_k \gtrsim_\sigma (k+1)^{-(1+\sigma)}$. For $\theta \in]0, 1/6[$ define

$$\mu_0 = \frac{1}{6} - \theta, \qquad \mu = \mu_0 \frac{1}{1+\sigma}.$$

Since $\rho_k \gtrsim \rho_0$ one has

$$\forall\, k < \min(k^*, \rho_0^{-\mu}), \qquad \begin{cases} q_n\delta_k \gtrsim \rho_0^{-1/6+\mu_0} = \rho_0^{-\theta} \\ (\rho_k\delta_k)^{-1} \lesssim \rho_0^{-1-\mu_0} = \rho_0^{-7/6+\theta} \\ q_n^2\delta_k^{-1}\rho_k^{1/2} \lesssim \rho_0^{-1/3+1/2-\mu_0} = \rho_0^{\theta}, \end{cases}$$

thus (note that for $k < \rho_0^{-\mu}$, $\rho_0 \ll_\theta 1$), $\delta_k^{-1}e^{-q_n\delta_k/2} \le \rho_0^{1/5}$ and consequently, if $\rho_0 \ll_\theta 1$,

$$\forall\, k < \min(k^*, \rho_0^{-\mu}), \quad \varepsilon_{k+1} \lesssim (\rho_0^{1/5} + \rho_0^{1/5} + \rho_0^{-1/3-7/3+2\theta}\sum_{j=0}^{k}\varepsilon_j + \rho_0^{\theta})\varepsilon_k$$

$$\le \overline{\mathrm{C}}_\theta \rho_0^{\theta}(1 + \rho_0^{-3}\sum_{j=0}^{k}\varepsilon_j)\varepsilon_k.$$

Since from (F.491) $\varepsilon_0 \le \rho_0^{2\overline{a}+1} < \rho_0^{1/5+(7/6)\overline{a}+1/3}$ we see that condition (F.483) of Lemma F.1 is satisfied (with $\alpha = \rho_0$, $\overline{k}_{\theta,\alpha} = \rho_0^{-\mu}$) hence

$$\text{(F.503)} \qquad \forall\, k \in [0, \rho_0^{-\mu}] \cap \mathbf{N}, \quad \varepsilon_k \le (2\overline{\mathrm{C}}_\theta\rho_0)^{\theta k}\varepsilon_0.$$

Now for any $0 < \beta \ll 1$, one can choose θ and σ so that $\rho_0^{-\mu} = q_n^{1-\beta}$ and in particular taking $\overline{k} = \overline{k}_\beta = [q_n^{1-\beta}]$ and using (F.501) one gets for $q_n \gg_\beta 1$

$$\text{(F.504)} \qquad \varepsilon_{\overline{k}} \le e^{-q_n^{1-\beta}}, \qquad s_{\overline{k}} \le 2\varepsilon_0$$

(we can assume $\overline{a} > 10$).

We now define

$$\mathrm{F}^{\mathrm{BNF}}_{q_n^{-1}} = \mathrm{F}_{\overline{k}}, \qquad \Omega^{\mathrm{BNF}}_{q_n^{-1}} = \Omega_{\overline{k}},$$

and

$$g^{\mathrm{BNF}}_{q_n^{-1}} = f^{-1}_{\mathrm{Y}_1^{\mathrm{W}h}} \circ \cdots \circ f^{-1}_{\mathrm{Y}_{\overline{k}}^{\mathrm{W}h}}$$

where Y_j^{Wh} is a C^2 Whitney extension of $(Y_j, e^{-\delta_j/2}W_j)$ given by Lemma 2.2; one has (*cf.* (F.493), (F.491))

$$\|g^{\mathrm{BNF}}_{q_n^{-1}} - id\|_{C^1} \lesssim q_n^2 \sum_{k=0}^{\overline{k}} (2\overline{C}_\theta \rho_0)^{\theta k} \varepsilon_0 \rho_0^{-4} h^{-4} k^8 \lesssim_{\theta,h} q_n^{-(m-26)}. \tag{F.505}$$

Inequalities (F.504) and (F.505), and the fact that $F_{\overline{k}} \in O^{\overline{k}+2m}(r)$, give the conclusion of Proposition 6.4. Note that (6.145) is a consequence of $F^{\mathrm{BNF}}_{q_n^{-1}} \in O^{q_n^{1-\beta}}(z,w)$ and Remark 6.2. For the last statement of the Proposition, we can choose $\Omega_{\overline{k}} = \Omega + \sum_{j=0}^{\overline{k}-1} M(F_j)^{Wh}$ where F_J^{Wh} is an C^3 Whitney extension of (F_j, W_j) given by Lemma 2.2. □

F.3 *Proof of Proposition 6.5 (BNF (AA) or (CC) Case, ω_0 Diophantine).* — The proof, that we mainly illustrate in the (AA)-Case, as well as the notations, are essentially the same as the ones of the proof of Proposition 6.4 (see Section F.2, in particular, we use the definitions (F.489), (F.490) for δ_k, ρ_k, W_k) with the following differences:

– we replace (F.488) by

$$\rho_0 = \rho^{(\tau+1)/\gamma}, \qquad W_0 = W_{h,\mathbf{D}(0,\rho_0)} = W_{h,\mathbf{D}(0,\rho^{b\tau})}$$

where $\gamma = 1$ in the (AA)-case and $\gamma = 1/2$ in the (CC)-case.

– since ω_0 is Diophantine, we can and do solve instead of (F.493) the equation without truncation (using Proposition 5.3 with $N = \infty$, $K = \kappa^{-1}$, *cf.* (6.149)):

$$-[Q_0]\cdot Y_k = F_k - \mathcal{M}_0(F_k), \qquad \|Y_k\|_{e^{-\delta_k/2}W_k} \lesssim \delta_k^{-\tau} \|F_k\|_{W_k}$$

where $Q_0(r) = 2\pi\omega_0 r$, cf. (F.494).

Notice that both in the (AA) and (CC) cases

$$F_0 = O^m(r), \qquad \|F_0\|_{W_0} \leq \rho_0^m. \tag{F.506}$$

In place of (F.496) we get (in the (AA)-case)

$$F_{k+1} = \dot{\mathfrak{O}}_2^{(\overline{a})}(Y_k, F_k) + O\Big(|\partial_\theta Y_k| \times |\partial(\Omega_k - Q_0)\circ r|\Big). \tag{F.507}$$

Since $F_k,\ Y_k \in O^{k+m}(r)(\theta)$ (see Remarks 2.1, 5.3) we have $O(|\partial_\theta Y_k| \times |\partial(\Omega_k - Q_0)\circ r|) = O^{k+m+1}(r)(\theta)$ and $\dot{\mathfrak{O}}_2^{(\overline{a})}(Y_k, F_k) = O^{2k+2m-\overline{a}}(r)(\theta)$ ($\overline{a}$ from Lemma 5.4). As a consequence, since $2m \geq \overline{a}$ we see that $F_{k+1} = O^{k+1+m}(r)(\theta)$.

From (F.507) and the fact that (*cf.* (2.54))

$$\|\partial_\theta Y_k\|_{e^{-\delta_k}W_k} \lesssim \delta_k^{-1} \|Y_k\|_{e^{-\delta_k/2}W_k}, \qquad \|\partial(\Omega_0 - Q_0)\circ r\|_{W_k} \leq \rho_k$$

hence

$$|\partial_\theta Y_k| \times |\partial(\Omega_0 - Q_0)\circ r| \lesssim \delta_k^{-(1+\tau)}\rho_k^\gamma \|F\|_{W_k}$$

where $\gamma = 1$ in the (AA)-case. A similar computation (*cf.* (F.498)) shows that one can take $\gamma = 1/2$ in the (CC)-case. With the notations $s_k = \|\Omega_k - \Omega_0\|_{\mathbf{D}(0,\rho_k)}$, $\varepsilon_k = \|F_k\|_{W_k}$, we then get

(**F.508**) $$\begin{cases} \varepsilon_{k+1} \lesssim \left((\rho_k\delta_k)^{-(\tau+\overline{a})}\varepsilon_k + (\rho_k\delta_k)^{-(2+\tau)}s_k + \delta_k^{-(1+\tau)}\rho_k^\gamma\right)\varepsilon_k \\ s_{k+1} \lesssim \sum_{j=0}^k \varepsilon_j \end{cases}$$

provided

(**F.509**) $$(\rho_k\delta_k)^{-(\tau+\overline{a})}\varepsilon_k < 1.$$

Let k^* be the largest integer for which the preceding sequences are defined and satisfies

(**F.510**) $$\forall k < k_*, \quad P_{\rho_0}(k,\varepsilon_k) := \rho_0^{-\gamma}(\rho_k\delta_k)^{-(\tau+\overline{a})}\varepsilon_k < 1;$$

the condition involved in (F.510) implies (F.509). From (F.489) for any $\sigma > 0$ one has $\delta_k \gtrsim_\sigma (k+1)^{-(1+\sigma)}$. Fix $\theta \in]0,\gamma[$ and define

$$\mu = \frac{\gamma-\theta}{1+\tau}\frac{1}{1+\sigma}.$$

Since $\rho_k \gtrsim \rho_0$ one has

$$\forall\ k < \min(k^*, \rho_0^{-\mu}), \qquad \begin{cases} (\rho_k\delta_k)^{-1} \lesssim \rho_0^{-1-\mu(1+\sigma)} \\ \delta_k^{-(1+\tau)}\rho_k^\gamma \lesssim \rho_0^{\gamma-(1+\tau)\mu(1+\sigma)} = \rho_0^\theta. \end{cases}$$

If we set

$$a = \left(1 + \frac{\gamma-\theta}{1+\tau}\right)(2+\tau) + \theta \le (3/2)(2+\tau) + 1$$

we then get using (F.510) and (F.508)

$$\begin{aligned} \forall\ k < \min(k^*, \rho_0^{-\mu}), \qquad \varepsilon_{k+1} &\le \overline{C}_\sigma\left(\rho_0^\gamma + \rho_0^{-a+\theta}\sum_{j=0}^k \varepsilon_j + \rho_0^\theta\right)\varepsilon_k \\ &\le \overline{C}_\sigma\rho_0^\theta \times \left(1 + \rho_0^{-a}\sum_{j=0}^k \varepsilon_j\right)\varepsilon_k. \end{aligned}$$

We now apply Lemma F.1 with $\alpha = \rho_0$: since ($\overline{a} > 10$)

$$\max\left(a, (1 + \frac{\gamma - \theta}{1+\tau})(\tau + \overline{a}) + 1\right) \leq 2(\tau + \overline{a}) \leq \overline{a}_{1,\tau} \leq m;$$

condition (F.506) shows that (F.510) is satisfied with $\overline{k} = \min(k^*, [\rho_0^{-\mu}])$ as well as conditions (F.483) for $\rho_0 \ll_{\theta,\sigma} 1$. We thus get if $\overline{k} := [\rho_0^{-\mu}]$,

$$(\mathbf{F.511}) \qquad \forall\, k \in [0, \overline{k}] \cap \mathbf{N}, \quad \varepsilon_k \leq \rho_0^{\theta k} \varepsilon_0.$$

Since θ and σ can be taken arbitrarily close to 1, for any $0 < \beta \ll 1$ one has for $\rho \ll_\beta 1$

$$\varepsilon_{\overline{k}} \leq e^{-\rho^{-(1-\beta)}}, \qquad s_{\overline{k}} \leq 2\overline{\varepsilon}_0.$$

We conclude like in the proof of Proposition 6.4 (Section F.2) by defining

$$F_\rho^{\mathrm{BNF}} = F_{\overline{k}}, \qquad \Omega_\rho^{\mathrm{BNF}} = \Omega_{\overline{k}}, \qquad g_\rho^{\mathrm{BNF}} = f^{-1}_{Y_1^{W_h}} \circ \cdots \circ f^{-1}_{Y_{\overline{k}}^{W_h}}.$$

Note that since $F_{\overline{k}} \in O^{\overline{k} + \overline{a}_{1,\tau}}(r)$ one has $F_\rho^{\mathrm{BNF}} \in O^{(1/\rho)^{1-\beta}}(r)$ and (6.154) is a consequence of Remark 6.4. □

Appendix G: Resonant Normal Forms

In this section we shall only consider the (AA)-Case.

Let $c \in \mathbf{D}(0,1)$, $\overline{\rho} > 0$, $h > 0$ and

$$(\mathbf{G.512}) \qquad \Omega \in \widetilde{\mathcal{O}}_\sigma(e^h \mathbf{D}(c, \overline{\rho})) \text{ and } F \in \mathcal{O}_\sigma(e^h W_{h, \mathbf{D}(c,\overline{\rho})})$$

where Ω satisfies the twist condition ($A, B \geq 1$)

$$\forall\, r \in \mathbf{R},\ A^{-1} \leq (2\pi)^{-1} \partial^2 \Omega(r) \leq A \text{ and } \|(2\pi)^{-1} D^3 \Omega\|_{\mathbf{C}} \leq B.$$

Our aim in this section is to give an approximate Normal Form for $\Phi_\Omega \circ f_F$ in a neighborhood of a q-resonant circle by which we mean that for some $(p, q) \in \mathbf{Z} \times \mathbf{N}^*$, $p \wedge q = 1$

$$(2\pi)^{-1} \partial \Omega(c) = \frac{p}{q}.$$

This Normal Form is quite similar in spirit (and in its construction) to the approximate BNF. It is used in the paper in Sections 8 (approximate Hamilton-Jacobi Normal Form) and 15 (creating hyperbolic periodic points).

As usual we define $\omega(c) := \frac{1}{2\pi} \partial \Omega(c)$.

Proposition **G.1** *(q-resonant Normal Form). — There exists a universal constant* $\overline{a}_3 \geq 10$ *(not depending on q and that we can assume in* **N***) such that, if one has*

$$(\textbf{G.513}) \qquad \begin{cases} \overline{\rho} < (\mathrm{A}q)^{-8} \\ \|\mathrm{F}\|_{\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} < \overline{\rho}^{\ \overline{a}_3}, \end{cases}$$

then the following holds: There exist $\overline{\Omega} \in \widetilde{\mathcal{O}}_\sigma(\mathbf{D}(c, e^{-1/q}\overline{\rho})) \cap \mathcal{TC}(2\mathrm{A}, 2\mathrm{B})$,

$$\overline{\mathrm{F}}^{res}, \mathrm{F}^{cor} \in \mathcal{O}_\sigma(e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}), \qquad g_{\mathrm{RNF}} \in \widetilde{\mathrm{Symp}}_{ex,\sigma}(e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})})$$

such that

$$(\textbf{G.514}) \qquad \begin{cases} g_{\mathrm{RNF}}^{-1} \circ \Phi_\Omega \circ f_{\mathrm{F}} \circ g_{\mathrm{RNF}} = \Phi_{2\pi(p/q)r} \circ \Phi_{\overline{\Omega}} \circ f_{\overline{\mathrm{F}}^{res}} \circ f_{\mathrm{F}^{cor}} \\ \overline{\mathrm{F}}^{res} \ \ is\ 2\pi/q - periodic, \qquad \mathcal{M}_0(\overline{\mathrm{F}}^{res}) = 0, \end{cases}$$

where

$$(\textbf{G.515}) \qquad \begin{cases} \|\overline{\Omega} - (\Omega - 2\pi(p/q)r)\|_{\mathbf{D}(c,e^{-1/q}\overline{\rho})} \lesssim \|\mathrm{F}\|_{\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \\ \|\overline{\mathrm{F}}^{res}\|_{e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim \|\mathrm{F}\|_{\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \\ \|g_{\mathrm{RNF}} - id\|_{\mathrm{C}^1} \lesssim (q\overline{\rho}^{\ -2})^2 \|\mathrm{F}\|_{h,\mathbf{D}(c,\overline{\rho})} \leq \overline{\rho}^{\ \overline{a}_3 - 5} \end{cases}$$

and

$$(\textbf{G.516}) \qquad \|\mathrm{F}^{cor}\|_{e^{-1/q}\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim \exp(-\overline{\rho}^{\ -1/4}) \|\mathrm{F}\|_{\mathrm{W}_{h,\mathbf{D}(c,\overline{\rho})}}.$$

We give the proof of this Proposition in the next subsections.

Remark **G.1.** — The implicit constants in the symbol $\lesssim$ of the preceding estimates depend on h; if $h_0 > 0$, they can be bounded above by a constant C_{h_0} whenever $h \geq h_0$.

G.1 *A preliminary Lemma.*

Lemma **G.2.** — *1) For any* $(k, l) \in \mathbf{Z}^* \times \mathbf{Z}$ *one has*

$$(\textbf{G.517}) \qquad \begin{cases} either\ q|k\ and\ p|l \\ or\ |k \times \frac{p}{q} - l| \geq \frac{1}{q}. \end{cases}$$

2) Let

$$\mathrm{N} = (q\overline{\rho}\mathrm{A})^{-1}.$$

For any $r \in \mathbf{D}(c, \overline{\rho})$ *and any* $(k, l) \in \mathbf{N}^* \times \mathbf{Z}$, $1 \leq k \leq \mathrm{N}$, *which is not in* $(q, p)\mathbf{Z}$ *one has*

$$(\textbf{G.518}) \qquad |k\omega(r) - l| \geq 1/(2q).$$

Proof. — 1) Indeed $|k(p/q)-l| = |kp-lq|/q$ and if the integer $kp-lq$ is 0 then $q|k$ and $p|l$.

2) We just notice that

$$\begin{aligned}|k\omega(r)-l| &\geq |k\omega(c)-l| - k|\omega(r)-\omega(c)|\\ &\geq (1/q) - N\|\partial\omega\|_{\mathbf{D}(c,\overline{\rho})}\overline{\rho}\\ &\geq 1/(2q).\end{aligned}$$

□

Define (see Section 5.1)

$$\text{(G.519)}\qquad T_N^{q-res}F = \sum_{\substack{k\in\mathbf{Z}\\ |k|<N\\ q|k}} \mathcal{M}_k(F), \qquad T_N^{q-nr}F = T_NF - T_N^{q-res}F.$$

We shall often use in what follows the shortcuts T_N^{res} and T_N^{nr} for T_N^{q-res}, T_N^{q-nr}.

From (5.101), (5.111) we see that

$$\text{(G.520)}\qquad T_N^{res}F\circ\phi_r^{2\pi/q} = T_N^{res}F$$

and

$$\text{(G.521)}\qquad \|T_N^{res}F\|_{e^{-\delta}h,\mathbf{D}(c,\overline{\rho})} \lesssim \delta^{-1}\|F\|_{h,\mathbf{D}(c,\overline{\rho})}.$$

Corollary **G.3.** — *For any* $F\in\mathcal{O}_\sigma(W_{h,\mathbf{D}(c,\overline{\rho})})$*, there exists* $Y\in\mathcal{O}_\sigma(W^{\Omega}_{h,\mathbf{D}(c,\overline{\rho})})$ *such that* $\mathcal{M}_0(Y)=0$ *and*

$$\text{(G.522)}\qquad T_N^{nr}F = [\Omega]\cdot Y.$$

This Y *satisfies for any* $0<\delta<h$

$$\text{(G.523)}\qquad \|Y\|_{e^{-\delta}W_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim q\delta^{-1}\|F\|_{W_{h,\mathbf{D}(c,\overline{\rho})}}.$$

Proof. — This is a simple adaptation of the proof of Proposition 5.3 (the non-resonance condition is replaced by (G.518)). □

G.2 *Elimination of non-resonant terms.*

Proposition **G.4.** — *There exists a universal constant* $\overline{a}_3$ *(not depending on* q*) such that if* $N=(q\overline{\rho}A)^{-1}$ *and*

$$\text{(G.524)}\qquad \begin{cases}\overline{\rho}^{1/8} < (qA)^{-1}\\ \|F\|_{W_{h,\mathbf{D}(c,\overline{\rho})}} < \overline{\rho}^{\,\overline{a}_3},\end{cases}$$

then there exist $F^{res}, F^{nr} \in \mathcal{O}_\sigma(e^{-1/q}W_{h,\mathbf{D}(0,\overline{\rho})})$, $g \in \widetilde{\mathrm{Symp}}_{ex,\sigma}(e^{-1/q}W_{h,\mathbf{D}(0,\overline{\rho})})$ *such that*

$$(\mathbf{G.525}) \qquad [e^{-1/q}W_{h,\mathbf{D}(0,\overline{\rho})}] \qquad g^{-1}\circ\Phi_\Omega\circ f_F\circ g = \Phi_\Omega\circ f_{F^{res}+F^{nr}}$$

$$(\mathbf{G.526}) \qquad F^{res} = T_N^{res}(F + O(q\overline{\rho}\|F\|_{W_{h,\mathbf{D}(c,\overline{\rho})}}))$$

F^{res} *being* $2\pi/q$*-periodic and*

$$(\mathbf{G.527}) \qquad \|g - id\|_{C^1} \lesssim (q\overline{\rho}^{-2})^2\|F\|_{h,\mathbf{D}(c,\overline{\rho})} \le \overline{\rho}^{\overline{a}_3-5}$$

$$(\mathbf{G.528}) \qquad \|F^{nr}\|_{e^{-1/q}W_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim q\exp(-\overline{\rho}^{-1/3})\|F\|_{h,\mathbf{D}(c,\overline{\rho})}.$$

Proof. — Note that we can assume, using Lemma 2.2 that $F \in \widetilde{\mathcal{O}}_\sigma(W_{h,\mathbf{D}(c,\overline{\rho})})$ and satisfies

$$(\mathbf{G.529}) \qquad \varepsilon_0 := \|F\|_{e^{-1/(10q)}W_{h,\mathbf{D}(c,\overline{\rho})}} \le \overline{\rho}^{\overline{a}_3}, \qquad \|F\|_{C^3} \lesssim \overline{\rho}^{\overline{a}_3-7}$$

where $\overline{a}_3$ will be defined in (G.539).

Let $N = (q\overline{\rho}A)^{-1}$. We define $W_0 = e^{-1/(10q)}W_{h,\mathbf{D}(c,\rho_0)}$, $\rho_0 = \overline{\rho}$ and for $k \ge 1$

$$(\mathbf{G.530}) \qquad \delta_k = \frac{(5q)^{-1}}{(k+1)^{4/3}}, \quad \rho_k = \exp(-\sum_{l=0}^{k-1}\delta_l)\overline{\rho}, \quad W_k = \exp(-\sum_{l=0}^{k-1}\delta_l)W_0$$

and we construct sequences $Y_k, F_k, F_k^{nr}, F_k^{res} \in \mathcal{O}_\sigma(W_k)$ such that $F_k = F_k^{nr} + F_k^{res}$

$$(\mathbf{G.531}) \qquad F_0^{res} = T_N^{res}F, \qquad F_0^{nr} = F - F_0^{res}, \qquad T_N^{res}F_0^{nr} = 0$$

and for $k \ge 0$

$$(\mathbf{G.532}) \qquad f_{Y_k}\circ\Phi_\Omega\circ f_{F_k^{nr}+F_k^{res}}\circ f_{Y_k}^{-1} = \Phi_\Omega\circ f_{F_{k+1}^{nr}+F_{k+1}^{res}}$$

where for any k,

$$F_k^{res}\circ\phi_{J\nabla r}^{2\pi/q} = F_k^{res}.$$

By Corollary G.3 there exists $Y_k \in \mathcal{O}_\sigma(e^{-\delta_k/2}W_k^\Omega)$ such that

$$(\mathbf{G.533}) \qquad [\Omega]\cdot Y_k = -T_N^{nr}F_k^{nr}, \qquad \|Y_k\|_{e^{-\delta_k/2}W_k^\Omega} \lesssim q\delta_k^{-1}\|F_k^{nr}\|_{W_k}.$$

Let $F_k := F_k^{res} + F_k^{nr}$ and compute using Proposition 4.7

$$\begin{aligned} f_{Y_k}\circ\Phi_\Omega\circ f_{F_k^{nr}+F_k^{res}}\circ f_{Y_k}^{-1} &= \Phi_\Omega\circ f_{F_k^{nr}+F_k^{res}+[\Omega]\cdot Y_k+\|DF_k\|_{e^{-\delta_k/2}W_k}\mathfrak{O}_1(Y_k)} \\ &= \Phi_\Omega\circ f_{R_N F_k^{nr}+T_N^{res}F_k^{nr}+F_k^{res}+q\delta_k^{-1}\|DF_k\|_{e^{-\delta_k/2}W_k}\mathfrak{O}_1(F_k^{nr})} \\ &= \Phi_\Omega\circ f_{F_{k+1}^{nr}+F_{k+1}^{res}} \end{aligned}$$

with

$$\textbf{(G.534)}\qquad \begin{cases} F_{k+1}^{nr} = R_N F_k^{nr} + q(\rho_k\delta_k)^{-2}\|F_k\|_{W_k}\mathfrak{O}_1(F_k^{nr}) \\ F_{k+1}^{res} = T_N^{res} F_k^{nr} + F_k^{res}. \end{cases}$$

In particular since $F_{k+1} = F_{k+1}^{nr} + F_{k+1}^{res} = F_{k+1}^{nr} + T_N^{res}F_k^{nr} + F_k^{res}$

$$\textbf{(G.535)}\qquad F_{k+1} = F_k + F_{k+1}^{nr} + T_N^{res}F_k^{nr} - F_k^{nr}.$$

If we define $\varepsilon_k^* = \|F_k^*\|_{W_k}$, $* = nr, res$, $\varepsilon_k = \|F_k\|_{W_k}$ we get from (G.531) and Lemma 5.2

$$\textbf{(G.536)}\qquad \varepsilon_0^{nr} \lesssim \delta_0^{-1}\varepsilon_0 \lesssim q\varepsilon_0$$

and from (G.534), (G.535) and Lemma 5.2 that for some $a > 0$

$$\textbf{(G.537)}\qquad \begin{cases} \varepsilon_{k+1}^{nr} \lesssim \delta_k^{-1}e^{-\delta_k N/2}\varepsilon_k^{nr} + q\varepsilon_k(\rho_k\delta_k)^{-a}\varepsilon_k^{nr} \\ \varepsilon_{k+1}^{res} \lesssim \varepsilon_k^{res} + \delta_k^{-1}\varepsilon_k^{nr} \\ \varepsilon_{k+1} = \varepsilon_k + O(\varepsilon_{k+1}^{nr} + \delta_k^{-1}\varepsilon_k^{nr}) \end{cases}$$

provided for some $a > 0$ (that we can assume ≥ 4)

$$\textbf{(G.538)}\qquad (\rho_k\delta_k)^{-a}\varepsilon_k^{nr} < 1.$$

From now on we define $\bar{a}_3 = 2a + 2$, see (G.529),

$$\textbf{(G.539)}\qquad \varepsilon_0 \leq \rho_0^{\bar{a}_3} \qquad \text{with} \quad \bar{a}_3 = 2a + 2 \geq 10;$$

notice that this implies (see (G.536))

$$\textbf{(G.540)}\qquad \varepsilon_0^{nr} \leq \rho_0^{2a}.$$

Let k^* be the largest integer for which the sequences $\varepsilon_k^{nr}, \varepsilon_k^{res}, \varepsilon_k$ are defined. We notice that for $k < \min(k^*, \rho_0^{-1/3})$ one has from (G.524)

$$(\rho_k\delta_k)^{-1} \lesssim \rho_0^{-1}q\rho_0^{-4/9} \lesssim A^{-1}\rho_0^{-1}\rho_0^{-1/8}\rho_0^{-4/9} \leq \rho_0^{-2}.$$

Since $\varepsilon_k = \varepsilon_0 + O(\delta_k^{-1}\sum_{j=0}^k \varepsilon_k^{nr})$ we get that for $k+1 \leq \rho_0^{-1/3}$, $\varepsilon_k = \varepsilon_0 + O(\rho_0^{-4/9}\sum_{j=0}^k \varepsilon_k^{nr})$ hence if $k < \min(k^*, \rho_0^{-1/3})$ (recall $q \leq \rho_0^{-1/8}$)

$$q\varepsilon_k(\rho_k\delta_k)^{-a}\varepsilon_k^{nr} \leq C\left(\rho_0^{-(2a+1)}\varepsilon_0 + \rho_0^{-(2a+2)}\sum_{j=0}^k \varepsilon_k^{nr}\right)\varepsilon_k^{nr}.$$

On the other hand, from (G.524), $q^{-1}N = q^{-2}\rho_0^{-1}A^{-1} \geq A^{-1}\rho_0^{-3/4}$ hence, if $k+1 \leq \rho_0^{-1/3}$ one has $\delta_k N = q^{-1}N/(k+1)^{4/3} \geq \rho_0^{-3/4}\rho_0^{4/9} = \rho_0^{-11/36}$ and thus $\delta_k^{-1}e^{-\delta_k N} \leq \rho_0$ if ρ_0

is small enough. The outcome of this is that for $k+1 \leq \rho_0^{-1/3}$ one has (we use condition (G.539))

$$\varepsilon_{k+1}^{nr} \leq C\rho_0 \left(1 + \rho_0^{-(2a+3)} \sum_{j=0}^{k} \varepsilon_k^{nr}\right) \varepsilon_k^{nr}.$$

Since for $\rho_0 \ll 1$ one has (*cf.* (G.540))

(G.541) $$\varepsilon_0^{nr} \leq \rho_0^{2a} \leq (\rho_0 \delta_0)^a$$

and we can thus apply Lemma F.1 with $\alpha = \rho_0$, to get

(G.542) $$k^* \geq \rho_0^{-1/3}, \qquad \forall 0 \leq k \leq k_*, \quad \varepsilon_k^{nr} \leq (2C\rho_0)^k \varepsilon_0^{nr} \leq e^{-k} q \varepsilon_0.$$

We now set

$$F^{res} = F_{k^*}^{res}, \qquad F^{nr} = F_{k^*}^{nr}, \qquad g = f_{Y_1^{Wh}}^{-1} \circ \cdots \circ f_{Y_{k^*-1}^{Wh}}$$

where Y_j^{Wh} is a C^2 Whitney extension of $(Y_j, e^{-\delta_j/2} W_j)$ given by Lemma 2.2. The conjugation relation (G.525) then holds and the conclusion (G.528) is satisfied since from (G.542)

$$e^{-2/q} W_{h,\mathbf{D}(c,\overline{\rho})} \subset W_{k^*}, \qquad \|F^{nr}\|_{e^{-2/q} W_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim e^{-\rho_0^{-1/3}} q \varepsilon_0.$$

To check (G.527) we just notice that from (G.533)

$$\|g - id\|_{C^1} \lesssim_h q^2 \rho_0^{-4} \varepsilon_0.$$

Finally, since $F^{res} = T_N^{res} F + T_N^{res}(\sum_{k=0}^{k_*} F_k^{nr})$ and $T_N^{res} F_0^{nr} = 0$ (*cf.* (G.531)) one has from the inequality $\varepsilon_k^{nr} \leq (2C\rho_0)^k \varepsilon_0^{nr} \lesssim \rho_0 e^{-(k-1)} \varepsilon_0$ ($\rho_0 \ll 1$, $k \geq 1$)

$$F^{res} = T_N^{res} F + T_N^{res} (\sum_{k=1}^{k_*} F_k^{nr}), \qquad \| \sum_{k=1}^{k_*} F_k^{nr} \|_{W_{k^*}} \leq \sum_{k=1}^{k^*} \varepsilon_k^{nr} \lesssim q \rho_0 \varepsilon_0$$

which gives conclusion (G.526):

$$F^{res} = T_N^{res} \Big(F + O(q \overline{\rho} \varepsilon_0) \Big).$$

□

G.3 *Proof of Proposition G.1.* — We apply Proposition G.4 and we write using Lemma 4.6

$$\begin{aligned} \Phi_\Omega \circ f_{F^{nr}+F^{res}} &= \Phi_{2\pi(p/q)(r-c)} \circ \Phi_{\Omega - 2\pi(p/q)(r-c)} \circ f_{F^{res}} \circ f_{F^{nr} + \|DF^{res}\|_{W_{h,U}} \mathfrak{O}_1(F^{nr})} \\ &= \Phi_{2\pi(p/q)(r-c)} \circ \Phi_{\Omega - 2\pi(p/q)(r-c)} \circ f_{F^{res}} \circ f_{F^{cor}} \end{aligned}$$

with

(**G.543**) $$\|F^{cor}\|_{e^{-1/q}W_{h,\mathbf{D}(c,\overline{\rho})}} \lesssim q^a\overline{\rho}^{\,-a}\|F^{nr}\|_{e^{-2/q}W_{h,\mathbf{D}(c,\overline{\rho})}}$$

provided for some $a > 0$

$$q^a\overline{\rho}^{\,-a}\|F^{nr}\|_{e^{-2/q}W_{h,\mathbf{D}(0,\overline{\rho})}} < 1.$$

The inequalities (G.524) and (G.528) show that this last condition is satisfied if $\overline{\rho} \ll 1$.

We now observe that

$$f_{F^{res}} = \Phi_{\mathcal{M}_0(F)} \circ f_{F^{res}-\mathcal{M}_0(F)}$$

and that

$$\Phi_{\Omega-2\pi(p/q)(r-c)} \circ \Phi_{\mathcal{M}_0(F)} = \Phi_{\Omega-2\pi(p/q)(r-c)+\mathcal{M}_0(F)}.$$

If we set

$$g_{\mathrm{RNF}} = g$$
$$\overline{\Omega} = \Omega - 2\pi(p/q)(r-c) + \mathcal{M}_0(F) \quad \text{and} \quad \overline{F}^{res} = F^{res} - \mathcal{M}_0(F)$$

and recall (G.525) we find the conjugation relation (G.514).

Note that since we have assumed that $F \in \widetilde{\mathcal{O}}_\sigma(e^{-1/(10q)}W_{h,\mathbf{D}(c,\overline{\rho})})$ satisfies (G.529) we have $\overline{\Omega} \in \widetilde{\mathcal{O}}_\sigma(e^{-1/q}W_{h,\mathbf{D}(c,\overline{\rho})}) \cap \mathcal{TC}(2A, 2B)$ and the first inequality of (G.515) is satisfied. The other inequalities of (G.515) are consequences of (G.526) and (G.527) and (G.516) is a consequence of (G.528), (G.524) and (G.543). □

Remark **G.2.** — Notice that from the first inequality of (5.111) in Lemma 5.2

(**G.544**) $$\|F^{res}\|_{W_{e^{-1/q}h,\mathbf{D}(c,e^{-1/q}\overline{\rho})}} \lesssim \|F\|_{W_{h,\mathbf{D}(c,\overline{\rho})}}.$$

Appendix H: Approximations by vector fields

The main result of this Section is the following proposition on the approximation of an exact symplectic diffeomorphism close to an integrable one by a vector field.

Proposition **H.1.** — *There exists a constant $\overline{C} > 0$ for which the following holds. Let $0 < \rho < 1$, $F \in \mathcal{O}_\sigma(\mathbf{T}_h \times D(0,\rho))$ and $\Omega \in \mathcal{O}_\sigma(\mathbf{D}(0,\rho))$, $\Omega(r) = O(r^2)$. If $\rho > 0$ is small enough, $h \gtrsim \rho^{1/3}$ and*

(**H.545**) $$\overline{C} \times (\rho h)^{-9}\|F\|_{h,\rho} < 1$$

then, there exist $\Pi \in \mathcal{O}_\sigma(\mathbf{D}(0,\rho/2))$, $A_3(F) \in \mathcal{O}_\sigma(\mathbf{T}_{h/2} \times D(0,\rho/2))$ *such that*

$$\Phi_\Omega \circ f_F = \Phi_\Pi \circ f_{A_3(F)} \tag{H.546}$$

with

$$\Pi = \Omega + F \circ \Phi_{-\Omega/2} + O(\rho^{1/4}\|F\|_{h,\rho}) \tag{H.547}$$

$$= \Omega + F + O(\rho^{1/4}\|F\|_{h,\rho}) \tag{H.548}$$

and

$$\|A_3(F)\|_{h/2,\rho/2} < \exp(-\rho^{-1/4})\|F\|_{h,\rho}. \tag{H.549}$$

The proof of this proposition is given in Section H.2.

H.1 *Auxiliary result.*

Proposition **H.2.** — *Let* $\rho > 0$, $\Omega(r) = O(r^2)$, $\Omega \in \mathcal{O}_\sigma(\mathbf{D}(0,\rho))$, $F, G \in \mathcal{O}_\sigma(\mathbf{T}_h \times \mathbf{D}(0,\rho))$ *such that (*C *some universal constant)*

$$C \times (\rho\delta)^{-4}(\|F\|_{h,\rho} + \|G\|_{h,\rho}) < 1. \tag{H.550}$$

Then for any $h/2 > \delta \gtrsim \rho^{1/3}$, *there exists* $A(F,G) \in \mathcal{O}_\sigma(e^{-\delta}(\mathbf{T}_h \times \mathbf{D}(0,\rho)))$ *such that*

$$\Phi_{\Omega+F+G} = \Phi_{\Omega+F} \circ \Phi_{G\circ\Phi_{\Omega/2}} \circ f_{A(F,G)} \tag{H.551}$$

with

$$\|A(F,G)\|_{h-\delta/2, e^{-\delta}\rho} \lesssim \left((\rho\delta)^{-4}(\|F\|_{h,\rho} + \|G\|_{h,\rho}) + \rho\delta^{-3}\right)\|G\|_{h,\rho}. \tag{H.552}$$

Proof. — To simplify the notations we denote $W = W_{h,\mathbf{D}(0,\rho)}$ and we assume that $\omega(r) := \nabla\Omega(r)$, $\omega(0) = 0$ satisfies

$$\omega(r) = r + O(r^2).$$

If

$$(\delta\rho)^{-2}\max(\|F\|_{h,\rho}, \|G\|_{h,\rho}) < 1$$

the images of the domain $e^{-2\delta}W$ by the flows Φ^t_Ω, $\Phi^t_{\Omega+F}$, $\Phi^t_{\Omega+F+G}$, $0 \le t \le 1$, are contained in $e^{-\delta}W$.

Let us denote

$$\sigma := \max(\|DF\|_{h,\delta}, \|DG\|_{h,\delta})$$

and for $x = (\theta, r) \in e^{-2\delta}W$ and $t \in [-1, 1]$

$$\Delta(t, x) = \Phi^t_{\Omega+F+G}(x) - \Phi^t_{\Omega+F}(x).$$

By classical theorems on ODE's for $t \in [-1, 1]$

$$\Delta(t, \cdot) = O(\sigma), \qquad \Phi^t_{\Omega+F} - \Phi^t_{\Omega} = O(\sigma).$$

On the other hand one has

$$\frac{d}{dt}\Delta(t, x) = J\nabla(\Omega + F + G) \circ \Phi^t_{\Omega+F+G}(x) - J\nabla(\Omega + F) \circ \Phi^t_{\Omega+F}(x)$$
$$= (I)(t, x) + (II)(t, x) + (III)(t, x) \tag{H.553}$$

with

$$(I)(t, x) = J\nabla\Omega \circ \Phi^t_{\Omega+F+G}(x) - J\nabla\Omega \circ \Phi^t_{\Omega+F}(x)$$

$$(II)(t, x) = J\nabla F \circ \Phi^t_{\Omega+F+G}(x) - J\nabla F \circ \Phi^t_{\Omega+F}(x)$$

$$(III)(t, x) = J\nabla G \circ \Phi^t_{\Omega+F+G}(x).$$

Since $\Phi^t_{\Omega+F+G}(x) = \Phi^t_{\Omega+F}(x) + (\Delta_\theta(t, x), \Delta_r(t, x))$ and $\Phi_{\Omega+F} - \Phi_\Omega = O(\sigma)$ one has (note that $r \circ \Phi^t_{\Omega+F} = r + O(\sigma)$)

$$(I)(t, x) = \begin{pmatrix} \omega(r \circ \Phi^t_{\Omega+F}(x) + \Delta_r(t, x)) - \omega(r \circ \Phi^t_{\Omega+F}(x)) \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \partial\omega(r)\Delta_r(t, x) + O(\sigma|\Delta_r(t, x)|) \\ 0 \end{pmatrix} \tag{H.554}$$

and

$$|(II)(t, x)| = O(|D^2F||\Delta(t, x)|). \tag{H.555}$$

We have using the fact that $\Phi^t_{\Omega+F+G} - \Phi_\Omega = O(\sigma)$ and $\omega(r) = O(r)$

$$(III)(t, x) = J\nabla G(\theta + t\omega(r), r) + O(\varepsilon\|D^2G\|)$$
$$= \begin{pmatrix} \partial_r G(\theta, r) + t\omega(r)\partial^2_{\theta r}G(\theta, r) + O(\rho^2\|\partial^2_\theta\partial_r G\|) \\ -\partial_\theta G(\theta, r) - t\omega(r)\partial^2_\theta G(\theta, r) + O(\rho^2\|\partial^3_\theta G\|) \end{pmatrix}$$
$$+ O(\sigma\|D^2G\|). \tag{H.556}$$

Summing (H.554), (H.555), (H.556) and integrating (H.553) gives

$$\textbf{(H.557)}\qquad \begin{pmatrix}\Delta_\theta(t,x)\\ \Delta_r(t,x)\end{pmatrix} = \begin{pmatrix}\partial\omega(r)\int_0^t \Delta_r(s,x)ds\\ 0\end{pmatrix} + \begin{pmatrix} t\partial_r G(\theta,r) + (t^2/2)\omega(r)\partial^2_{\theta r}G(\theta,r)\\ -t\partial_\theta G(\theta,r) - (t^2/2)\omega(r)\partial^2_\theta G(\theta,r)\end{pmatrix} + O(\varepsilon + |D^2F|)\int_0^t |\Delta(s,x)|ds + A_1$$

with

$$A_1 = O(\sigma\|D^2G\|) + O(\rho^2\|D\partial^2_\theta G\|).$$

Lemma **H.3.** — *One has*

$$|\Delta(t,x)| \le A_2 := O(\|DG\| + \rho\|D\partial_\theta G\|) + O(\sigma\|D^2G\|) + O(\rho^2\|D\partial^2_\theta G\|).$$

Proof. — From (H.557) and the fact that $\partial\omega(r) \asymp 1$

$$|\Delta(t,x)| \le C(1+\varepsilon+\|D^2F\|_{h,\rho})\int_0^t |\Delta(s,x)|ds + O(\|DG\| + \rho\|D\partial_\theta G\|) + A_1$$

and we conclude by Grönwall inequality. □

Looking at the second component of (H.557) gives ($\omega(r) = O(r)$)

$$\Delta_r(t,x) = -t\partial_\theta G(\theta,r) + O(\rho\|\partial^2_\theta G\|) + O((\varepsilon + \|D^2F\|)A_2) + A_1$$

hence (integrating again and putting the result in (H.557))

$$\textbf{(H.558)}\qquad \begin{pmatrix}\Delta_\theta(t,x)\\ \Delta_r(t,x)\end{pmatrix} = \begin{pmatrix} -\partial\omega(r)(t^2/2)\partial_\theta G(\theta,r) + t\partial_r G(\theta,r) + (t^2/2)\omega(r)\partial^2_{\theta r}G(\theta,r)\\ -t\partial_\theta G(\theta,r) - (t^2/2)\omega(r)\partial^2_\theta G(\theta,r)\end{pmatrix} + O(A_3)$$

with

$$A_3 = O(\rho\|\partial^2_\theta G\|) + O((\varepsilon + \|D^2F\|)A_2) + A_1.$$

Taking $t = 1$ gives

$$\Phi_{\Omega+F+G}(x) = \Phi_{\Omega+F}(x)$$

$$+\begin{pmatrix} -(\partial\omega(r)/2)\partial_\theta G(\theta,r)+\partial_r G(\theta,r)+(\omega(r)/2)\partial^2_{\theta r}G(\theta,r) \\ -\partial_\theta G(\theta,r)-(\omega(r)/2)\partial^2_\theta G(\theta,r)\end{pmatrix}$$

$$+O(A_3).$$

On the other hand

$$\begin{pmatrix} \partial_r(G(\theta-\omega(r)/2,r)) \\ -\partial_\theta(G(\theta-\omega(r)/2,r))\end{pmatrix} = \begin{pmatrix} -(\partial\omega(r)/2)\partial_\theta G(\theta-\omega(r)/2,r)+\partial_r G(\theta-\omega(r)/2,r) \\ -\partial_\theta G(\theta-\omega(r)/2,r)\end{pmatrix}$$

hence

$$J\nabla(G\circ\Phi_{-\Omega/2})\circ\Phi_\Omega = \begin{pmatrix} -(\partial\omega(r)/2)\partial_\theta G(\theta+\omega(r)/2,r)+\partial_r G(\theta+\omega(r)/2,r) \\ -\partial_\theta G(\theta+\omega(r)/2,r)\end{pmatrix}$$

and from Taylor Formula and the fact that $\omega(r)=O(r)$

$$\Phi_{\Omega+F+G}(x)-\Phi_{\Omega+F}(x)=J\nabla(G\circ\Phi_{-\Omega/2})\circ\Phi_\Omega+O(\rho\|\partial^2_\theta G\|)+O(A_3).$$

Since $\Phi_{\Omega+F}=\Phi_\Omega+O(\sigma)$, this means that

$$\Phi_{\Omega+F+G}=(id+J\nabla(G\circ\Phi_{-\Omega/2})\circ\Phi_{\Omega+F}+O(\sigma\|D^2G\|)+O(A_3)$$

thus

$$\Phi_{\Omega+F+G}=\Phi_{G\circ\Phi_{-\Omega/2}}\circ\Phi_{\Omega+F}+O(A_3)$$

or

$$\Phi_{\Omega+F+G}=f_{O(A_3)}\circ\Phi_{G\circ\Phi_{-\Omega/2}}\circ\Phi_{\Omega+F}$$

with

$$\begin{aligned} A_3 &\lesssim ((\rho\delta)^{-2}\sigma+\rho^2(\rho\delta)^{-1}\delta^{-2}+(\sigma+(\rho\delta)^{-2}\sigma) \\ &\quad\times((\rho\delta)^{-1}+\rho(\rho\delta)^{-1}\delta^{-1})+\rho\delta^{-2})\|G\|_{h,\rho} \\ &\lesssim((\rho\delta)^{-3}\sigma+\rho\delta^{-3})\|G\|_{h,\rho} \\ &\lesssim\left((\rho\delta)^{-4}(\|F\|_{h,\rho}+\|G\|_{h,\rho})+\rho\delta^{-3}\right)\|G\|_{h,\rho}\end{aligned}$$

provided

$$(\rho\delta)^{-4}(\|F\|_{h,\rho}+\|G\|_{h,\delta})<1,\qquad \rho\delta^{-3}<1.$$

To conclude we observe that if we apply the preceding formula with $-\Omega$ and $-F$ instead of Ω, F

$$\Phi_{-\Omega-F-G} = f_{O(A_3)} \circ \Phi_{-G\circ\Phi_{\Omega/2}} \circ \Phi_{-\Omega-F}$$

and inverting

$$\Phi_{\Omega+F+G} = \Phi_{\Omega+F} \circ \Phi_{G\circ\Phi_{\Omega/2}} + f_{O(A_3)}. \qquad \square$$

Corollary **H.4.** — *Under the same conditions of Proposition H.4 one has*

$$\Phi_{\Omega+F} \circ f_G = \Phi_{\Omega+F+G\circ\Phi_{-\Omega/2}} \circ f_{A_2(F,G)}$$

with

$$\|A_2(F,G)\|_{h-\delta/2, e^{-\delta}\rho} \lesssim \Big((\rho\delta)^{-4}(\|F\|_{h,\rho} + \|G\|_{h,\rho}) + \rho\delta^{-3}\Big)\|G\|_{h,\rho}. \tag{H.559}$$

Proof. — If we apply (H.551) with $G \circ \Phi_{-\Omega/2}$ instead of G we get

$$\Phi_{\Omega+F+G\circ\Phi_{-\Omega/2}} = \Phi_{\Omega+F} \circ \Phi_G \circ f_{A(F,G\circ\Phi_{-\Omega/2})}$$

hence

$$\begin{aligned}\Phi_{\Omega+F} \circ f_G &= \Phi_{\Omega+F+G\circ\Phi_{-\Omega/2}} \circ f^{-1}_{A(F,G\circ\Phi_{-\Omega/2})} \circ \Phi_G^{-1} \circ f_G \\ &= \Phi_{\Omega+F+G\circ\Phi_{-\Omega/2}} \circ f_{A_2(F,G)}\end{aligned}$$

where $A_2(F,G) = A(F, G \circ \Phi_{-\Omega/2}) + O(|DG||D^2G|)$ satisfies (H.559) (*cf.* (H.552)). $\square$

H.2 *Proof of Proposition H.1.* — Let $\delta_k = c/(k+1)^{3/2}$, $h_k = h - \delta_k/2$, $\rho_0 = (3/4)\rho$, $\rho_k = e^{-\delta_k}\rho$ and c chosen such that $h_k \geq h/2$, $\rho_k \geq \rho/2$ for all $k \in \mathbf{N}$. Using Corollary H.4 we construct sequences S_k, G_k such that $S_0 = 0$, $G_0 = F$

$$\Phi_{\Omega+S_k} \circ f_{G_k} = \Phi_{\Omega+S_{k+1}} \circ f_{G_{k+1}} \tag{H.560}$$

$$\begin{cases} S_{k+1} = S_k + G_k \circ \Phi_{-\Omega/2} \\ G_{k+1} = A_2(S_k, G_k) \end{cases} \tag{H.561}$$

with

$$\|S_{k+1}\|_{h_{k+1},\rho_{k+1}} \leq \|S_k\|_{h_k,\rho_k} + \|G_k\|_{h_k,\rho_k}$$

and

$$\|G_{k+1}\|_{h_{k+1},\rho_{k+1}} \lesssim \rho_k\delta_k^{-3}\|G_k\|_{h_k,\rho_k} + (\rho_k\delta_k)^{-4}(\|S_k\|_{h_k,\rho_k} + \|G_k\|_{h_k,\rho_k})\|G_k\|_{h_k,\rho_k} \tag{H.562}$$

as long as

$$(\rho_k\delta_k)^{-4}(\|S_k\|_{h_k,\rho_k} + \|G_k\|_{h_k,\rho_k}) < 1.$$

With $\varepsilon_n = \|G_n\|_{h_n,\rho_n}$ and $\sigma_n := \|S_n\|_{h_n,\rho_n}$ we have ($s_0 = 0$)

$$\varepsilon_{k+1} \le C(\rho_k\delta_k^{-3} + (\rho_k\delta_k)^{-4}\sum_{j=0}^{k}\varepsilon_j)\varepsilon_k \tag{H.563}$$

$$\sigma_{k+1} \le \sigma_k + O(\varepsilon_k) \tag{H.564}$$

as long as $(\rho_k\delta_k)^{-4}(s_k + \sigma_k) < 1$.

Let k^* be the largest integer for which these sequences are defined. We observe that for $k < \min(k^*, \rho^{1/4})$ one has $(\rho_k\delta_k)^{-1} \le \rho^{-2}$ and $\rho_k\delta_k^{-3} \le \rho_k^{1-3/4} = \rho^{1/4}$; hence, if $\overline{k} = \min(k^*, \rho^{-1/4})$ one has

$$\forall\ k < \overline{k}, \qquad \varepsilon_{k+1} \le C\rho^{1/4}(1 + \rho^{-9}\sum_{j=0}^{k}\varepsilon_j)\varepsilon_k.$$

We are in position to apply Lemma F.1 with $\alpha = \rho$, $\theta = 1/4$, $a = 9$: since condition (F.483) is satisfied (*cf.* (H.545)) one has

$$k^* \ge \rho^{1/4}, \qquad \forall\ k \le k^*,\ \ \varepsilon_k \le (2C\rho)^{k/4}\varepsilon_0$$

and also

$$s_k - \varepsilon_0 \le \sum_{j=1}^{k}\varepsilon_j \lesssim \rho^{1/4}\varepsilon_0.$$

To conclude the proof we set

$$\Pi = S_{\overline{k}}, \qquad A_3(F) = F_{\overline{k}}.$$ □

Appendix I: Adapted KAM domains: lemmas

I.1 *Proof of Lemma 10.1.*

I.1.1 *Proof of the RHS inequality of (10.306).* — From (7.193) and the definition (10.300) of $i_-(\rho)$, for every $(k, l) \in E_{i_-(\rho)-1}$, $0 < k < N_{i_-(\rho)-1}$, $0 \le |l| \le N_{i_-(\rho)-1}$ one has

$$\mathbf{D}(c_{l/k}^{(i_-(\rho)-1)}, K_{i_-(\rho)-1}^{-1}) \cap \mathbf{D}(0, 2\rho) = \emptyset,$$

hence $|c_{l/k}^{(i_-(\rho)-1)}| > \rho$. Since $\omega_{i_-(\rho)-1}(c_{l/k}^{(i_-(\rho)-1)}) = l/k$, we deduce from the fact that Ω_i satisfies a (2A, 2B)-twist condition (7.166) that $|(l/k) - \omega_0| = |\omega(c_{l/k}^{(i_-(\rho)-1)}) - \omega(0)| \ge$

$(2A)^{-1}\rho$. By Dirichlet Approximation Theorem, for any $L \geq 1$ there exist $k, l \in \mathbf{Z}$, $0 < |k| \leq L$ such that $|\omega_0 - (l/k)| \leq \frac{1}{|k|L}$, hence

$$(2A)^{-1}\rho \leq \frac{1}{|k|L}. \tag{I.565}$$

However, since $\omega_0 \in DC(\tau)$ one has $\frac{1}{|k|^{1+\tau}} \lesssim |\omega_0 - (l/k)| \leq \frac{1}{|k|L}$ hence

$$L^{1/\tau} \lesssim |k|.$$

This and (I.565) yield

$$\rho \lesssim L^{-(1+1/\tau)}.$$

In particular if one chooses $L = N_{i_-(\rho)-1} - 1 \asymp N_{i_-(\rho)}$ one gets

$$\rho \lesssim N_{i_-(\rho)}^{-(1+1/\tau)}$$

which proves the inequality of the RHS of (10.306).

I.1.2 *Proof of the LHS inequality of (10.306).* — Let us prove the second inequality of (10.306). By definition of $i_-(\rho)$ there exists $(l, k) \in \mathbf{Z}^2$, $0 < k < N_{i_-(\rho)}$, $|l| \leq N_{i_-(\rho)}$ such that (*cf.* (7.193))

$$\mathbf{D}(c_{l/k}^{(i_-(\rho))}, 2K_{i_-(\rho)}^{-1}) \cap \mathbf{D}(0, 2\rho) \neq \emptyset.$$

In particular (*cf.* (7.163)) $|c_{l/k}^{(i_-(\rho))}| \leq 3\rho$. Since $\omega_0 \in DC(\kappa, \tau)$, $|\omega(0) - (k/l)| \geq \kappa/k^{1+\tau}$ and from (7.192) $|\omega_{i_-(\rho)}(0) - \omega_{i_-(\rho)}(c_{l/k}^{(i_-(\rho))})| \geq \kappa/k^{1+\tau} - 2\bar{\varepsilon}^{1/2}$; by the twist condition $6A\rho \geq 2A|c_{l/k}^{(i_-(\rho))}| \geq \kappa N_{i_-(\rho)}^{-(1+\tau)} - \rho^2$ hence

$$\rho \gtrsim N_{i_-(\rho)}^{-(1+\tau)}$$

which shows that the LHS of (10.306) holds.

Estimates (10.304), (10.305) are then immediate. □

I.2 *Proof of Items 1, 2, 4 of Proposition 10.2.* — Recall that from (10.307)

$$\begin{cases} U_i^{((3/2)\rho)} = \mathbf{D}(0, (3/2)\rho) \setminus \bigcup_{j=1}^{i-1} \bigcup_{(k,l)\in E_j} \mathbf{D}(c_{l/k}^{(j)}, s_{j,i-1}K_j^{-1}), \\ s_{j,i-1} = e^{\sum_{m=j}^{i-1}\delta_m} \in [1, 2] \end{cases} \tag{I.566}$$

where $E_j \subset \{(k, l) \in \mathbf{Z}^2,\ 0 < k < N_j,\ 0 < |l| \leq N_j\}$, $\omega_j(c_{l/k}^{(j)}) = l/k$. In particular, any $D \in \mathcal{D}(U_i)$ is of the form $D = \mathbf{D}(c_{l/k}^{(j)}, s_{j,i-1}K_j^{-1})$, where $j \leq i-1$, $(k, l) \in E_j$.

Lemma **I.1.** — *If* $D \in \mathcal{D}_{(3/2)\rho}(U_i)$ *then* $j \geq i_-(\rho)$.

Proof. — Since $\mathbf{D}(0,2\rho)=\mathbf{D}(0,2\rho)\cap \mathrm{U}_{i_-(\rho)}$, from (I.566) for all $j\le i_-(\rho)-1$, $(k,l)\in \mathrm{E}_j$ one has $|c^{(j)}_{l/k}|\ge 2\rho+\mathrm{K}_j^{-1}$. On the other hand, if $\mathrm{D}\in\mathcal{D}(\mathrm{U}_i)$ is of the form $\mathrm{D}=\mathbf{D}(c^{(j)}_{l/k},s_{j,i-1}\mathrm{K}_j^{-1})$, where $j\le i-1$, $(k,l)\in\mathrm{E}_j$ and intersects $\mathbf{D}(0,(3/2)\rho)$ one has $|c^{(j)}_{l/k}|\le(3/2)\rho+2\mathrm{K}_j^{-1}<2\rho+\mathrm{K}_j^{-1}$ hence $j\ge i_-(\rho)$. □

From (7.193) and Lemma I.1 we can thus write

$$\textbf{(I.567)}\qquad \begin{cases}\mathrm{U}_i^{((3/2)\rho)}=\mathbf{D}(0,(3/2)\rho)\setminus\bigcup_{j=i_-(\rho)}^{i-1}\bigcup_{(k,l)\in\mathrm{E}_j}\mathbf{D}(c^{(j)}_{l/k},s_{j,i-1}\mathrm{K}_j^{-1}),\\ s_{j,i-1}=e^{\sum_{m=j}^{i-1}\delta_m}\in[1,2].\end{cases}$$

We define

$$\mathrm{Q}_i=\bigcup_{j=i_-(\rho)}^{i-1}\{l/k,\ (k,l)\in\mathrm{E}_j\}$$

and for $t\in\mathrm{Q}_i$

$$j(t,i)=\min\{j:\ j\in\mathbf{N}\cap[i_-(\rho),i-1],\ (k,l)\in\mathrm{E}_j\text{ and }l/k=t\}$$

$$c(t,i)=c_t^{(j(t,i))},\qquad s(t,i)=s_{j(t,i),i-1}.$$

Define for $i_-(\rho)\le j<i\le i_+(\rho)$,

$$\kappa_{j,i}=s_{j,i-1}\mathrm{K}_j^{-1}.$$

We observe that from the inequality $\mathrm{N}_{i_+(\rho)}\le\mathrm{N}^2_{i_-(\rho)}$, for any $i_-(\rho)\le j<i\le i_+(\rho)$, $i_-(\rho)\le j'<i'\le i_+(\rho)$, $(i,j)\ne(i',j')$, one has[46]

$$\kappa_{j,i}+\kappa_{j',i'}\ll\mathrm{N}^{-2}_{\max(j,j')},\qquad \overline{\varepsilon}^{\,1/2}_{\min(j,j')}\ll|\kappa_{j,i}-\kappa_{j',i'}|;$$

if $j'<j$ or $j'=j$ and $i<i'$, one has $\kappa_{j,i}<\kappa_{j',i'}$ hence from Lemma 7.3, Item (2) for $(k,l)\in\mathrm{E}_j$, $(k',l')\in\mathrm{E}_{j'}$ one has

$$\textbf{(I.568)}\qquad \begin{cases}\text{either } l/k\ne l'/k' \quad\text{and}\quad \mathbf{D}(c^{(j)}_{l/k},\kappa_{j,i})\cap\mathbf{D}(c^{(j')}_{l'/k'},\kappa_{j',i'})=\emptyset\\ \text{or } l/k=l'/k' \quad\text{and}\quad \mathbf{D}(c^{(j)}_{l/k},\kappa_{j,i})\subset\mathbf{D}(c^{(j')}_{l'/k'},\kappa_{j',i'}).\end{cases}$$

As a consequence, for $j\in\mathbf{N}\cap[i_-(\rho),i-1]$, $(k,l)\in\mathrm{E}_j$, $t=l/k$, one has the inclusion $\mathbf{D}(c^{(j)}_{l/k},s_{j,i-1}\mathrm{K}_j^{-1})\subset\mathbf{D}(c(t,i),s_{j(t,i),i-1}\mathrm{K}^{-1}_{j(t,i)})$, and therefore (*cf.* (I.567), (10.308))

$$\mathrm{U}_i^{((3/2)\rho)}=\mathbf{D}(0,(3/2)\rho)\setminus\bigcup_{t\in\mathrm{Q}_i}\mathbf{D}(c(t,i),s_{j(t,i),i-1}\mathrm{K}^{-1}_{j(t,i)}).$$

[46] This is clear if $j\ne j'$; if $j=j'$ observe that if $i\ne i'$, $\overline{\varepsilon}^{\,1/2}_{i_-(\rho)}\ll|s_{j,i-1}-s_{j,i'-1}|\mathrm{K}_j^{-1}$.

This implies that any $\mathrm{D} \in \mathcal{D}_{(3/2)\rho}(\mathrm{U}_i)$ is of the form

$$(\mathbf{I.569}) \qquad \mathrm{D} = \mathbf{D}(c(t,i), s_{j(t,i),i-1}\mathrm{K}_{j(t,i)}^{-1}), \qquad t \in \mathrm{Q}_j, \quad j(t,i) \le i-1.$$

Proof of item 1 of Proposition 10.2. — This is a consequence of (I.569) and (I.568). □

Proof of item 2 of Proposition 10.2. — One can write for some $t \in \mathrm{Q}_j$, $t' \in \mathrm{Q}_{j'}$, $\mathrm{D} = \mathbf{D}(c(t,i), s_{j(t,i),i-1}\mathrm{K}_{j(t,i)}^{-1})$, $\mathrm{D}' = \mathbf{D}(c(t',i'), s_{j(t',i'),i'-1}\mathrm{K}_{j(t',i')}^{-1})$ and from Lemma 7.3, Item (2) if $\mathrm{D} \cap \mathrm{D}' \neq \emptyset$ one has $t = t'$. On the other hand since $t = t' \in \mathrm{Q}_{j'} \subset \mathrm{Q}_j$ one has $j(t,i') = j(t,i)$. We now use the fact that $s_{j(t,i'),i'-1} \le s_{j(t,i),i-1}$. □

Proof of item 4 of Proposition 10.2. — Let us prove that $\mathrm{D} \in \mathcal{D}_\rho(\mathrm{U}_{i_+(\rho)})$ is a subset of $\mathrm{U}_{i_\mathrm{D}}$. If this were not the case, there would exist $\mathrm{D}' \in \mathcal{D}(\mathrm{U}_{i_\mathrm{D}})$ such that $\mathrm{D}' \cap \mathrm{D} \neq \emptyset$; in particular $\mathrm{D}' \in \mathcal{D}_{(3/2)\rho}(\mathrm{U}_{i_\mathrm{D}})$ and from item 2 $\mathrm{D}' \subset \mathrm{D}$; but this contradicts the definition of i_D. Hence $\mathrm{D} \subset \mathrm{U}_{i_\mathrm{D}}$.

This latter inclusion and (I.567) applied with $i = i_\mathrm{D}$ show that one has $\mathrm{D} \cap \mathbf{D}(c_{l/k}^{(j)}, s_{j,i-1}\mathrm{K}_j^{-1}) = \emptyset$ for all $i_-(\rho) \le j \le i_\mathrm{D} - 1$, $(k,l) \in \mathrm{E}_j$. As a consequence $\mathrm{D} = \mathbf{D}(c_{l/k}^{(j)}, s_{j,i_+(\rho)-1}\mathrm{K}_j^{-1})$ for some $j \ge i_\mathrm{D}$, $(k,l) \in \mathrm{E}_j$.

On the other hand, by definition of i_D there exists $\mathrm{D}' \in \mathcal{D}_\rho(\mathrm{U}_{i_\mathrm{D}+1})$ of the form $\mathrm{D}' = \mathbf{D}(c', s'\mathrm{K}_{j'}^{-1})$ with $j' \le i_\mathrm{D}$, $s' \in [1,2]$ (*cf.* (I.569)) such that $\mathrm{D}' \subset \mathrm{D}$. One hence have $s_{j,i_+(\rho)-1}\mathrm{K}_j^{-1} \ge \mathrm{K}_{i_\mathrm{D}}^{-1}$ thus $j \ge i_\mathrm{D}$. We conclude that $j = i_\mathrm{D}$. □

Appendix J: Classical KAM measure estimates

J.1 *A lemma.* — If A and B are two sets we denote by $\mathrm{A}\triangle\mathrm{B} = (\mathrm{A} \cup \mathrm{B}) \setminus (\mathrm{A} \cap \mathrm{B})$ their symmetric difference.

Lemma **J.1.** — *Let* $\mathrm{A} = \mathrm{I} \setminus \bigcup_{j\in\mathrm{J}} \mathrm{I}_j$, *where* $\mathrm{I} \subset \mathbf{R}$ *is an interval and all the intervals* I_j *are disjoint. Then if* $\sum_{j\in\mathrm{J}} |\mathrm{I}_j|^{1/2} \le 1$ *and if* $g : \mathrm{M}_\mathbf{R} \to \mathrm{M}_\mathbf{R}$ *is a* C^1*-symplectic diffeomorphism such that* $\|g - id\|_{\mathrm{C}^1} \le 1/10$, *then one has* $\mathrm{Leb}(\mathrm{W}_\mathrm{A} \triangle \mathrm{W}_{g(\mathrm{A})}) \lesssim \|g - id\|_{\mathrm{C}^0}^{1/2}$.

Proof. — We can assume that the intervals I_j are contained in I. Recall that $\mathbf{1}_{\mathrm{A}\,\triangle\,\mathrm{B}} = |\mathbf{1}_\mathrm{A} - \mathbf{1}_\mathrm{B}|$ and notice that since the intervals I_j are pairwise disjoint one has $\mathbf{1}_{\mathrm{W}_\mathrm{A}} = \mathbf{1}_{\mathrm{W}_\mathrm{I}} - \sum_{j\in\mathrm{J}} \mathbf{1}_{\mathrm{W}_{\mathrm{I}_j}}$ hence

$$\mathbf{1}_{\mathrm{W}_\mathrm{A}\,\triangle\,g(\mathrm{W}_\mathrm{A})} = \left|\chi - \sum_{j\in\mathrm{J}} \chi_j\right|$$

where $\chi = \mathbf{1}_{W_I} - \mathbf{1}_{g(W_I)}$, $\chi_j = \mathbf{1}_{W_{I_j}} - \mathbf{1}_{g(W_{I_j})}$. This gives

$$\begin{aligned}
&\mathrm{Leb}_{M_{\mathbf{R}}}(W_A \bigtriangleup g(W_A)) \\
&\quad = \|\chi - \sum_{j\in J} \chi_j\|_{L^1} \\
&\quad \le \|\chi\|_{L^1} + \sum_{j\in J} \|\chi_j\|_{L^1} \\
&\quad \le \mathrm{Leb}_{M_{\mathbf{R}}}(W_I \bigtriangleup g(W_I)) + \sum_{j\in J} \mathrm{Leb}_{M_{\mathbf{R}}}(W_{I_j} \bigtriangleup g(W_{I_j})).
\end{aligned}$$

On the other hand, if I is an interval, there exist intervals $\check{I} \subset I \subset \widehat{I}$ such that $W_{\check{I}} \subset g(W_I) \subset W_{\widehat{I}}$ and $\max(|I \bigtriangleup \widehat{I}|, |I \bigtriangleup \check{I}|) \le 2\max(\|g - id\|_{C^0}, \|g^{-1} - id\|_{C^0}) \le C\|g - id\|_{C^0}$, $C > 0$ depending only on M (recall that we have assumed $\|g - id\|_{C^1}$ is small enough). This is clear in the (AA)-case and in the (CC) or (CC*)-case it follows from the (AA)-case using the symplectic changes of coordinates $\psi_\pm$ and φ (4.79), (4.77). Therefore, since g is symplectic,

$$\mathrm{Leb}_{M_{\mathbf{R}}}(W_{I_j} \bigtriangleup g(W_{I_j})) \le C\min(\|g - id\|_{C^0}, \mathrm{Leb}_{M_{\mathbf{R}}}(W_{I_j})).$$

In particular $\mathrm{Leb}_{M_{\mathbf{R}}}(W_{I_j} \bigtriangleup g(W_{I_j})) \le C\|g - id\|_{C^0}^{1/2} \mathrm{Leb}_{M_{\mathbf{R}}}(W_{I_j})^{1/2}$ and since $\mathrm{Leb}_{M_{\mathbf{R}}}(W_{I_j}) \le |I_j|$ the conclusion follows. □

J.2 *Proof of Theorem 12.1.* — We use the notations of Section 7 and Propositions 7.1, 7.2, 7.5 and Remark 7.1.

We apply Proposition 7.5–Remark 7.1 with $m = 1$ and Proposition 4.3 with $A = e^{-2\delta_1}U$, $L = L_{1,\mathrm{Prop.}\ 7.5}$, $\widetilde{A} = \overline{U_1} = \overline{U}$,

$$\begin{aligned}
&\mathrm{Leb}_{M_{\mathbf{R}}}(W_{\mathbf{R}\cap e^{-2\delta_1}U} \setminus \mathcal{L}(f, W_{\mathbf{R}\cap\overline{U}})) \\
&\quad \le C \times (\mathrm{Leb}_{\mathbf{R}}(\mathbf{R} \cap (e^{-2\delta_1}U \setminus L)) + \|g_{1,\infty} - id\|_{C^0}^{1/2}) \\
&\quad \lesssim \overline{\varepsilon}^{\frac{1}{2(\overline{a}_0+3)}} + \overline{\varepsilon}^{1/8} \lesssim \overline{\varepsilon}^{\frac{1}{2(\overline{a}_0+3)}}.
\end{aligned}$$

□

Appendix K: From (CC) to (AA) coordinates

Sometimes we need to reduce the (CC)-case to the (AA)-case, for example when defining the Hamilton-Jacobi Normal Form in Section 8 or in Section 16.

For $\alpha \in]0, \pi[$ define the angular sector $\Delta_\alpha^+(\rho) = \{r \in \mathbf{D}(0,\rho),\ \arg(r) \notin [-\alpha, \alpha]\}$ and $\Delta_\alpha^-(\rho) = -\Delta_\alpha^+(\rho)$. Recall the definition of the maps $\psi_\pm$, *cf.* (4.79) of Section 4.1.

Lemma **K.1.** — *Let* $c \in \mathbf{R}$, $F^{CC} \in \mathcal{O}_\sigma(W^{CC}_{h,\mathbf{D}(c,2\rho)})$, $\overline{\varepsilon} = Ce^{h/2}\|D^2F^{CC}\|_{W^{CC}_{h,\mathbf{D}(c,2\rho)}}$,

$$C\delta^{-2}\rho^{-2}\overline{\varepsilon} < 1 \tag{K.570}$$

and $\alpha \in]\delta, \pi - \delta[$.

(1) *If* $c = 0$ *and* $F^{CC} = O^3(z, w)$ *there exists* $F^{AA}_\pm \in \mathcal{O}_\sigma(\mathbf{T}_{h-\delta} \times \Delta^\pm_{\alpha+4\delta}(\rho - 4\delta))$ *such that on* $\mathbf{T}_{h-4\delta} \times \Delta^\pm_{\alpha+4\delta}(\rho - 4\delta)$ *one has*

$$f_{F^{AA}_\pm} = \psi_\pm^{-1} \circ f_{F^{CC}} \circ \psi_\pm, \qquad F^{AA}_\pm = F^{CC} \circ \psi_\pm + \mathfrak{O}_2(F^{CC}). \tag{K.571}$$

(2) *If* $|c| > 4\rho$, *there exists* $F^{AA}_\pm \in \mathcal{O}_\sigma(\mathbf{T}_{h-\delta} \times \mathbf{D}(0, \rho))$ *such that* (K.571) *holds.*

Proof. — We prove item (1), the proof of item (2) is done in a similar (and even simpler) way. From $\overline{\varepsilon} \le \delta$ and $f_F(0) = 0$ we get that if z, w satisfy $|z|, |w| < e^{h-\delta}(\rho - 3\delta)^{1/2}$, $r = -izw \in \Delta^\pm_{\alpha+3\delta}(\rho - 3\delta)$ then $\widetilde{z}, \widetilde{w}$ defined as $(\widetilde{z}, \widetilde{w}) = f_{F^{CC}}(z, w)$ satisfy $|\widetilde{z} - z| \le e^{-h/2}\overline{\varepsilon}|z|$, $|\widetilde{w} - w| \le e^{-h/2}\overline{\varepsilon}|w|$ and thus $|\widetilde{z}| \le e^{\overline{\varepsilon}}|z| \le e^h\rho^{1/2}$ and $|\widetilde{w}| \le e^{\overline{\varepsilon}}|w| \le e^h\rho^{1/2}$; on the other hand if $\widetilde{r} = -i\widetilde{z}\widetilde{w}$ one has

$$|\widetilde{r} - r| \le 3\overline{\varepsilon}|r|, \qquad |\arg(\widetilde{r}) - \arg(r)| \le 3\overline{\varepsilon}, \qquad \left|\frac{\widetilde{z}/\widetilde{w}}{z/w} - 1\right| \le (5/2)\overline{\varepsilon}. \tag{K.572}$$

Since $\overline{\varepsilon} < \delta$,

$$f_{F^{CC}} \circ \psi_\pm(\mathbf{T}_{h-3\delta} \times \Delta^\pm_{\alpha+3\delta}(0, \rho - 3\delta)) \subset \psi_\pm(\mathbf{T}_h \times \Delta^\pm_\alpha(0, \rho))$$

hence $f^{AA} := \psi_\pm^{-1} \circ f_{F^{CC}} \circ \psi_\pm : \mathbf{T}_{h-3\delta} \times \Delta^\pm_{\alpha+3\delta}(0, \rho - 3\delta) \to \mathbf{T}_h \times \Delta^\pm_\alpha(0, \rho)$ is well defined. On the other hand if $\psi_\pm^{-1}(z, w) = (\theta, r)$, $f^{AA}_\pm(\theta, r) = (\widetilde{\theta}, \widetilde{r})$, $\psi_\pm(\widetilde{\theta}, \widetilde{r}) = (\widetilde{z}, \widetilde{w})$ one has from (K.572) and Lemma M.1

$$\max(|\widetilde{\theta} - \theta|_{2\pi\mathbf{Z}}, |\widetilde{r} - r|) \le 3\overline{\varepsilon}$$

hence

$$\|f^{AA} - id\|_{\mathbf{T}_{h-3\delta} \times \Delta^\pm_{\alpha+3\delta}(0,\rho-3\delta)} \le 3\overline{\varepsilon}$$

and from Remark 4.2, Lemmata 4.4, 4.5 and condition (K.570) there exists $F^{AA}_\pm \in \mathcal{O}(\mathbf{T}_{h-4\delta} \times \Delta^\pm_{\alpha+4\delta}(0, \rho - 4\delta))$ such that $f_{F^{AA}_\pm} = f^{AA}$ and

$$f_{F^{AA}_\pm} = \phi^1_{J\nabla F^{AA}_\pm} \circ f_{\mathfrak{O}_2(F^{AA}_\pm)}. \tag{K.573}$$

To get the second estimate in (K.571) we notice that

$$f_{F^{CC}} = \phi^1_{J\nabla F^{CC}} \circ f_{\mathfrak{O}_2(F^{CC})}$$

hence

$$f_{F^{AA}_{\pm}} = \psi_{\pm}^{-1} \circ \phi^1_{J\nabla F^{CC}} \circ f_{\mathfrak{O}_2(F^{CC})} \circ \psi_{\pm}$$
$$= \phi^1_{J\nabla(F^{CC}\circ\psi_{\pm})} \circ \psi_{\pm}^{-1} \circ f_{\mathfrak{O}_2(F^{CC})} \circ \psi_{\pm}$$
$$= \phi^1_{J\nabla(F^{CC}\circ\psi_{\pm})} \circ f_{\mathfrak{O}_2(F^{CC})}$$

and from (K.573)

$$F^{AA}_{\pm} = F^{CC} \circ \psi_{\pm} + \mathfrak{O}_2(F^{CC}). \qquad \square$$

Remark **K.1.** — If $f^{CC} = \Phi^{CC}_{\Omega} \circ f_{F^{CC}}$ we have (*cf.* Section 4.2).

$$\psi_{\pm}^{-1} \circ f^{CC} \circ \psi_{\pm} = \Phi^{AA}_{\Omega} \circ f_{F^{AA}_{\pm}}.$$

Appendix L: Lemmas for Hamilton-Jacobi Normal Forms

L.1 *Proof of Lemma 8.3.* — Since $\partial_r^2 \widetilde{\Omega}(r) \asymp 1$ (*cf.* (8.218)), (8.222) and (8.220) show that there exists $e_0 : \mathbf{T}_{qh/3} \to \mathbf{C}$, $e_0 \in \mathcal{O}_\sigma(\mathbf{T}_{qh/3})$ such that

(**L.574**) $$\forall\, \theta \in \mathbf{T}_{qh/3},\ \partial_r \widetilde{\Pi}(\theta, e_0(\theta)) = 0, \qquad \|e_0\|_{\mathbf{T}_{qh/3}} \lesssim (q\overline{\rho})^{-1} q^2 \overline{\varepsilon}.$$

We now make a Taylor expansion: using (L.574) we see that

(**L.575**) $$\begin{aligned}\widetilde{\Pi}(\theta, r) &= \widetilde{\Pi}(\theta, e_0(\theta) + (r - e_0(\theta)) \\ &= \widetilde{\Pi}(\theta, e_0(\theta)) + (1/2)\partial_r^2 \widetilde{\Pi}(\theta, e_0(\theta))(r - e_0(\theta))^2 \\ &\quad + (r - e_0(\theta))^3 \sum_{k=3}^{\infty} \frac{1}{k!} \partial_r^k \widetilde{\Pi}(\theta, e_0(\theta))(r - e_0(\theta))^{k-3}\end{aligned}$$

and if we define

(**L.576**) $$\varpi(\theta) = (1/2)\partial_r^2 \widetilde{\Pi}(\theta, e_0(\theta)), \qquad e_1(\theta) = -\widetilde{\Pi}(\theta, e_0(\theta))/\varpi(\theta)$$

one gets for some $p(\theta, r)$

$$\begin{aligned}\widetilde{\Pi}(\theta, r) &= \varpi(\theta)\Big(-e_1(\theta) + (r - e_0(\theta))^2 + (r - e_0(\theta))^3 p(\theta, r)\Big) \\ &= \overline{\Pi}(\theta, r - e_0(\theta))\end{aligned}$$

with $\overline{\Pi} \in \mathcal{O}(\mathbf{T}_{qh/3} \times \mathbf{D}(0, e^{-2/q} q\overline{\rho}/2 - Cq\overline{\rho}^{\ -1}\overline{\varepsilon})) \subset \mathcal{O}(\mathbf{T}_{qh/3} \times \mathbf{D}(0, \rho_q))$

$$\overline{\Pi}(\theta, r) = \varpi(\theta)\Big(r^2 - e_1(\theta) + r^3 p(\theta, r + e_0(\theta))\Big);$$

this gives the desired form for $\overline{\Pi}(\theta, r)$ if one sets $f(\theta, r) = p(\theta, r + e_0(\theta))$.

The estimates (8.228) on e_0, e_1, ϖ are then clear from (L.574), (L.576). Let us check the one on f. From (8.223) and (8.227) we have

$$\varpi(\theta)(r^2 - e_1(\theta) + r^3 f(\theta, r))$$
$$=: \varpi r^2 + \sum_{i=0}^{2} f_i(\theta)(r + e_0(\theta))^i + r^3(b(r) + \widetilde{f}(\theta, r + e_0(\theta)))$$

hence from (8.225) and the first two inequalities of (8.228)

$$r^3\Big(f(\theta, r) - \varpi(\theta)^{-1}\Big(b(r) - \widetilde{f}(\theta, r + e_0(\theta))\Big)\Big) \lesssim (q\overline{\rho})^{-3} q^2 \overline{\varepsilon}$$

and by the maximum principle

$$\sup_{(\theta,r)\in \mathbf{T}_{qh/3}\times \mathbf{D}(0,\rho_q)} \Big| f(\theta, r) - \varpi(\theta)^{-1}\Big(b(r) - \widetilde{f}(\theta, r + e_0(\theta))\Big)\Big|$$
$$\lesssim \overline{\rho}^{-3}(q\overline{\rho})^{-3} q^2 \overline{\varepsilon} \ll 1.$$

We the conclude by (8.218) and (8.225). □

L.2 *Square roots.*

Lemma **L.1.** — *Let $a \in \mathbf{C}^*$. There exists a unique function $m_a(z) = z(1 + a/z^2)^{1/2}$ univalent on $\mathbf{C} \smallsetminus \overline{\mathbf{D}}(0, |a|^{1/2})$ such that*

(**L.577**) $$m_a^2(z) = z^2 + a, \qquad m_a(z) = z + \mathrm{O}(z^{-1}).$$

It satisfies for $z, z' \in \mathrm{E_L} := \{w \in \mathbf{C}, |w| > \mathrm{L}|a|^{1/2}\}$ ($\mathrm{L} > 3$)

(**L.578**) $$(2/\pi)e^{-2/\mathrm{L}^2} \leq \left|\frac{m_a(z) - m_a(z')}{z - z'}\right| \leq (\pi/2)e^{1/\mathrm{L}^2}$$

Proof. — The existence and uniqueness of $m_a(z) = z(1 + (a/z^2))^{1/2}$ is clear.

Note that the inverse for the composition of m_a is m_{-a} and that if $\mathrm{L} > 2$ $m_a(\mathrm{E_L}) \subset \mathrm{E}_{3\mathrm{L}/4}$. On the other hand the derivative of $m_a(z)$ is equal to $\partial_z m_a(z) = (1 + a/z^2)^{-1/2}$ and since for $t \in [0, 1/2]$, $(1 - t)^{-1/2} \leq 1 + t$ one gets for $z \in \mathrm{E_L}$ ($\mathrm{L} > 2$) $|\partial_z m_a(z)| \leq e^{1/\mathrm{L}^2}$. Now any two points $z, z' \in \mathrm{E_L}$ can be joined by a path in $\mathrm{E_L}$ the length of which is $\leq (\pi/2)|z - z'|$; thus for any $z, z' \in \mathrm{E_L}$, $|m_a(z) - m_a(z')| \leq (\pi/2)e^{1/\mathrm{L}^2}|z - z'|$ which is the right hand side inequality of (L.578). To get the left hand side we use the fact that $|m_{-a}(m_a(z)) - m_{-a}(m_a(z'))| \leq (\pi/2)e^{1/(3\mathrm{L}/4)^2}|m_a(z) - m_a(z')|$ if $\mathrm{L} > 3$ ($3\mathrm{L}/4 > 2$). □

L.3 *Proof of Lemma 8.5.* — From Lemma L.1 $z \mapsto (z^2+a)^{1/2}$ is well defined on $\mathbf{C} \setminus \{|z| > |a|^{1/2}\}$.

Let $0 \le s \le h/3$. We are looking for $g(\theta, z) = \varpi(\theta)^{-1/2} z(1+\mathring{g}(\theta, z))$ such that

$$z^2 = \varpi(\theta)\Big(z^2\varpi(\theta)^{-1}(1+\mathring{g}(\theta,z))^2 - e_1(\theta) + z^3\varpi(\theta)^{-3/2}(1+\mathring{g}(\theta,z))^3 f(\theta, g(\theta,z))\Big)$$

which can be written as a Fixed Point problem

$$\mathring{g}(\theta,z) = \left(1+\varpi(\theta)\frac{e_1(\theta)}{z^2} - z\varpi(\theta)^{-1/2}(1+\mathring{g}(\theta,z))^3 f(\theta, g(\theta,z))\right)^{1/2} - 1. \tag{L.579}$$

Using the estimate on f given by (8.228) one can see that the map $\Psi : \mathring{g} \mapsto$ R.H.S. of (L.579) defines a $2\rho_q$-contracting map on the ball $B(0, CL^{-2})$ of center 0 and radius CL^{-2} of the Banach space $(\mathcal{O}(\mathbf{T}_{sq} \times \mathbf{A}(\lambda_{s,L}, \rho_q)), \|\cdot\|_\infty)$ provided L^{-1} and ρ_q are small enough. By the Contraction Mapping Theorem it has a unique fixed point $\mathring{g}$ in this ball. In other words

$$\overline{\Pi}(\theta, g(\theta,z)) = z^2. \tag{L.580}$$

The fact that $g \in \mathcal{O}(\mathbf{T}_{qs} \times \mathbf{A}(\lambda_{s,L}, \rho_q))$ is uniquely defined shows that the various g found for different values of s must agree. Hence g is defined on $\bigcup_{0\le s\le 1}(\mathbf{T}_{qs} \times \mathbf{A}(\lambda_{s,L}, \rho_q))$. □

L.4 *Proof of Lemma 8.6.* — We look for H under the form $H(z) = \gamma^{-1} z(1+\mathring{H}(z))$. Equation (8.238) can be written as a Fixed Point problem:

$$\mathring{H}(z) = -\frac{\mathring{\Gamma}(\gamma^{-1}z(1+\mathring{H}(z))))}{(1+\mathring{\Gamma}(\gamma^{-1}z(1+\mathring{H}(z))))}. \tag{L.581}$$

By Cauchy's estimates for $z \in \mathbf{A}(\lambda_{s,L}, \rho_q)$

$$|\partial\mathring{\Gamma}(z)| \le \frac{1}{\operatorname{dist}(z, \partial\mathbf{A}(\lambda_{s,L}, \rho_q))} L^{-2}.$$

Hence if $z \in \mathbf{A}(2\lambda_{s,L}, (1/2)\rho_q)$ the map $u \mapsto \mathring{\Gamma}(\gamma^{-1}zu)$ is $4L^{-2}$-Lipschitz on $\{(3/4) \le |u| \le 4/3\}$ and the map Ψ defined by the R.H.S. of (L.581) is $4L^{-2}$ contracting on the ball $\{\|\mathring{H}\|_{\mathbf{A}(2\lambda_{s,L},(1/2)\rho_q)} \le 2L^{-2}\}$. It admits thus a unique fixed point in this ball. □

Appendix M: Some other lemmas

Lemma **M.1.** — *Let* $z \in \mathbf{C}$

1) One has

$$|e^{iz}-1| \geq \frac{1}{2}\min(1, \min_{l\in\mathbf{Z}}|z-2\pi l|).$$

2) If $z\in\mathbf{R}$,

$$|e^{iz}-1| \geq \frac{2}{\pi}\min_{l\in\mathbf{Z}}|z-2\pi l|).$$

Proof. — 1) Let $\eta := e^{iz}-1$. We can assume $|\eta|<1/2$. We can thus define $iz_0 := \ln(1+\eta) = \sum_{k\in\mathbf{N}^*}(-1)^{k-1}\eta^k/k$ such that $e^{iz_0} = 1+\eta = e^{iz}$. There thus exists $l\in\mathbf{Z}$ such that $z_0 = z-2\pi l$. But $|z_0| = |\ln(1+\eta)| \leq 2|\eta|$.

2) Just use the fact that for $|w|\leq\pi$, $|2\sin(w/2)|\geq(2/\pi)|w|$. □

Lemma **M.2.** — *Let* $f\in C_h^{\omega}(\mathbf{T})$ *be such that for some* $\delta\in]0,1[$, $\mu>0$

(**M.582**) $$\|f\|_{L^2(\mathbf{T})} \leq \delta\|f\|_{C^0(\mathbf{T})} + \mu.$$

Then, for some $C>0$,

(**M.583**) $$\|f\|_{C^0(\mathbf{T})} \leq \delta^{-1}\mu + \frac{C}{h}e^{-h/(12\delta^2)}\|f\|_h.$$

Proof. — If

(**M.584**) $$f(\theta) = \sum_{k\in\mathbf{Z}}\widehat{f}(k)e^{iik\theta}$$

is the Fourier expansion of f, one has for some $C>0$ and any $N\in\mathbf{N}^*$

(**M.585**) $$\|f\|_{C^0(T)} \leq \sum_{|k|\leq N}|\widehat{f}(k)| + \frac{C}{h}e^{-hN}\|f\|_h$$

(**M.586**) $$\leq (2N+1)^{1/2}\|f\|_{L^2(\mathbf{T})} + \frac{C}{h}e^{-hN}\|f\|_h$$

(**M.587**) $$\leq (3N)^{1/2}(\delta\|f\|_{C^0(\mathbf{T})}+\mu) + \frac{C}{h}e^{-hN}\|f\|_h.$$

If we choose $N=\delta^{-2}/12$ we have $(3N)^{1/2}\delta\leq 1/2$ and

(**M.588**) $$\|f\|_{C^0(\mathbf{T})} \leq \delta^{-1}\mu + \frac{C}{h}e^{-h/(12\delta^2)}\|f\|_h.$$

Appendix N: Stable and unstable Manifolds

N.1 *The Stable Manifold Theorem.* — Let $(E, \|\cdot\|)$ be a Banach space, $M : E \to E$ an invertible linear continuous map. Let $\kappa, \delta > 0$. We say that M is (κ, δ)-*hyperbolic* if there exist $\kappa > 0$ and continuous projectors P_s, P_u satisfying $id_E = P_s + P_u$, $P_sP_u = P_uP_s = 0$, $P_sMP_u = P_uMP_s = 0$ such that

$$\begin{cases} \max(\|P_sMP_s\|, \|(P_uMP_u)^{-1}\|) \leq e^{-\kappa} \\ \max(\|P_s\|, \|P_u\|) \leq \delta^{-1}. \end{cases}$$

The spaces $E_* := P_*E$, $* = s, u$, are then M-invariant and are the stable and unstable spaces of the linear map M. We shall use the notations $M_* = P_*MP_*$, $* = s, u$.

Let $B(0, \rho) \subset E$ be the ball of center 0 and radius $\rho > 0$.

Theorem **N.1** *(Stable/Unstable Manifold Theorem). — Assume that* M *is* (κ, δ)*-hyperbolic as above and let* $F : B(0, \rho) \to E$ *be* C^1. *Assume that*

$$\|F(0)\| \leq C^{-1}\delta\kappa\rho, \quad \|DF\|_{C^1(B(0,\rho))} \leq C^{-1}\delta\kappa. \tag{N.589}$$

Then, if C *is large enough (but universal)*

(1) The map $x \mapsto Mx + F(x)$ *has a unique hyperbolic fixed point* $\bar{x}$ *such that* $\max(\|P_s\bar{x}\|, \|P_u\bar{x}\|) \leq (\rho/4)$ *(in particular, it is located in* $B(0, \rho/2)$*).*

(2) The local stable (resp. unstable) manifold

$$W^s_{loc}(\bar{x}; M + F) := \{y \in B(\bar{x}, \rho/4),\ \forall\, n \geq 0,\ (M + F)^n(y) \in B(\bar{x}, \rho/2)\}$$

(resp. $W^u_{loc}(\bar{x}; M + F) := \{y \in B(\bar{x}, \rho/4),\ \forall\, n \leq 0,\ (M + F)^n(y) \in B(\bar{x}, \rho/2)\}$*) of the point* $\bar{x}$ *for* $M + F$ *is of the form* $\{x_s + \gamma_{s,F}(x_s),\ x_s \in E_s \cap B(0, \rho/2)\}$ *(resp.* $\{x_u + \gamma_{u,F}(x_u),\ x_u \in E_u \cap B(0, \rho/2)\}$*) where* $\gamma_{s,F} : E_s \to E_u$ *(resp.* $\gamma_{u,F} : E_u \to E_s$*) is a map of class* C^1 *and* $\|D\gamma_{s,F}\| \leq C\|DF\|_{B(0,\rho)}(\delta\kappa)^{-1}$ *(resp.* $\|D\gamma_{u,F}\| \leq C\|DF\|_{B(0,\rho)}(\delta\kappa)^{-1}$*).*

(3) If G *satisfies also (N.589) then for* $* = s, u$, $\|D\gamma_{*,F} - D\gamma_{*,G}\| \leq C\|D(F - G)\|_{B(0,\rho)}(\delta\kappa)^{-1}$.

(4) If $F(0) = 0$, $DF(0) = 0$ *then* $\bar{x} = 0$ *and* $T_0W^*_{loc}(0) = E_*$, $* = s, u$.

Notice that the Theorem gives the same size for the domains of definition of $\gamma_{s,F}$, $\gamma_{u,F}$.

N.2 *Proof of Lemma 15.4.* — Using the definition of $f_{H_Q+\widetilde{\omega}}(\theta, r) = (\varphi, R)$ one can see that $f_{H_Q+\widetilde{\omega}}(\theta, r) = (\theta, r)$ if and only if

$$\nabla H_Q(\theta, r) + \nabla\widetilde{\omega}(r) = 0 \tag{N.590}$$

or equivalently

$$0 = \widetilde{a}(0)\theta + \partial_r\widetilde{b}(0)r$$

$$0 = \partial_r\widetilde{b}(0)\theta + (\varpi + \partial_r^2\widetilde{a}(0))r + \partial_r\widetilde{a}(0) + \partial_r\widetilde{\omega}(r).$$

Solving the first equation and inserting it into the second yields

$$\text{(N.591)} \qquad \theta = -\frac{\partial_r \widetilde{b}(0)}{\widetilde{a}(0)} r$$

$$\text{(N.592)} \qquad r = -\frac{\partial_r \widetilde{a}(0) + \partial_r \widetilde{\omega}(r)}{\varpi + \partial_r^2 \widetilde{a}(0) - \frac{(\partial_r \widetilde{b}(0))^2}{\widetilde{a}(0)}}.$$

We observe that, *cf.* (15.422),

$$\max(|\partial_r^2 \widetilde{a}(0)|, |(\partial \widetilde{b}_r(0))^2 / \widetilde{a}(0)|) \lesssim q^2 \overline{\nu}_q^{-1} \rho_{p/q}^{-2} e^{-qh} \varepsilon_{p/q} < \varpi / 10$$

$$|\partial_r \widetilde{a}(0)| \lesssim q^2 \rho_{p/q}^{-1} e^{-qh} \varepsilon_{p/q}, \qquad \partial_r \widetilde{\omega}(r) = \mathrm{O}(r^2)$$

and deduce by a simple fixed point theorem (in dimension 1) that (N.592) has a unique real solution $r_0 \asymp \partial_r \widetilde{a}(0)$; returning to (N.591) and using (*cf.* (15.423), (15.422))

$$\text{(N.593)} \qquad \widetilde{a}(0) = q^2 \overline{\nu}_q e^{-qh} \varepsilon_{p/q}, \qquad |\partial_r \widetilde{b}(0)| \lesssim q^2 \rho_{p/q}^{-1} e^{-qh} \varepsilon_{p/q}$$

we conclude that (N.590) has also a unique solution $(\theta_0, r_0) \in \mathbf{D}(0, \rho_{p/q})^2$

$$|\theta_0| \lesssim \overline{\nu}_q^{-1} q^2 \rho_{p/q}^{-2} \varepsilon_{p/q} e^{-qh}, \qquad |r_0| \lesssim q^2 \rho_{p/q}^{-1} \varepsilon_{p/q} e^{-qh}$$

and in particular since $\rho_{p/q}^{-8} = \max((c_{p/q}/4)^{-8}, q^{72})$ (*cf.* (15.405)), $q^2 e^{-qh} = \mathrm{O}(q^{-100})$, $\varepsilon_{p/q} \leq c_{p/q}^{\overline{a}_3}$ (*cf.* (15.406)), $\nu_q \gtrsim q \rho_{p/q}$ (*cf.* (15.409)), $\overline{a}_3 \geq 10$, one has

$$\text{(N.594)} \qquad (\theta_0, r_0) \in (\mathbf{D}(0, \rho_{p/q}^5) \times \mathbf{D}(0, \rho_{p/q}^5)) \cap \mathbf{R}^2.$$

We now compute $\mathrm{D} f_{\mathrm{H_Q} + \widetilde{\omega}}(\theta_0, r_0)$. Since $\widetilde{\omega}$ depends only on the r-variable one has, *cf.* (4.95) of Lemma 4.6,

$$f_{\mathrm{H_Q} + \widetilde{\omega}} = \Phi_{\widetilde{\omega}} \circ f_{\mathrm{H_Q}}$$

hence

$$\mathrm{D} f_{\mathrm{H_Q} + \widetilde{\omega}}(\theta_0, r_0) = \begin{pmatrix} 1 & \partial_r^2 \widetilde{\omega}(r_0) \\ 0 & 1 \end{pmatrix} \mathrm{D} f_{\mathrm{H_Q}}.$$

A simple computation shows that the derivative of the symplectic map $f_{\mathrm{H_Q}}$ is equal to

$$\mathrm{D} f_{\mathrm{H_Q}} = \begin{pmatrix} 1 + \partial \widetilde{b}(0) + \frac{(\varpi + \partial_r^2 \widetilde{a}(0)) \widetilde{a}(0)}{1 + \partial \widetilde{b}(0)} & \frac{\varpi + \partial_r^2 \widetilde{a}(0)}{1 + \partial \widetilde{b}(0)} \\ \frac{\widetilde{a}(0)}{1 + \partial \widetilde{b}(0)} & \frac{1}{1 + \partial \widetilde{b}(0)} \end{pmatrix}$$

hence

$$\operatorname{tr}(Df_{H_Q+\tilde\omega}) = 2 + \varpi\tilde a(0)(1 + O(q^2\rho_{p/q}^{-2}\varepsilon_{p/q}e^{-qh})) + O((\partial_r\tilde b(0))^2) + \frac{\partial^2\tilde\omega(r_0)\tilde a(0)}{1+\partial\tilde b(0)}.$$

The estimate (N.594) on r_0, the fact that $\partial^2\tilde\omega(r_0) = O(r_0)$ and (N.593) show that

$$\operatorname{tr}(Df_{H_Q+\tilde\omega}) = 2 + \varpi q^2\overline{\nu}_q\varepsilon_{p/q}e^{-qh}(1 + O(\rho_{p/q}^5)).$$

Since $Df_{H_Q+\tilde\omega}(\theta_0, r_0) \in SL(2,\mathbf{R})$ we deduce that it is a (κ,δ)-hyperbolic matrix with

(N.595) $$\delta = \kappa = q(\varpi\nu_q\varepsilon_{p/q}e^{-qh})^{1/2}(1 + o_{1/q}(1))$$

(we used that $\overline{\nu}_q = \nu_q(1 + o_{1/q}(1))$, *cf.* (15.416)).

The statement on the eigendirections is then a simple computation. □

REFERENCES

1. V. I. Arnold, Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian, *Usp. Mat. Nauk*, **18** (1963), 13–40.
2. V. I. Arnold, V. V. Kozlov and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*, Springer, Berlin, 1997, 291 pp.
3. A. Avila, J. De Simoi and V. Kaloshin, An integrable deformation of an ellipse of small eccentricity is an ellipse, *Ann. Math. (2)*, **184** (2016), 527–558.
4. G. D. Birkhoff, Surface transformations and their dynamical applications, *Acta Math.*, **43** (1922), 1–119.
5. G. D. Birkhoff, *Dynamical Systems*, AMS, Providence, 1927.
6. A. Bounemoura, Non-degenerate Liouville tori are KAM stable, *Adv. Math.*, **292** (2016), 42–51.
7. A. D. Brjuno, Analytical form of differential equations I, II, *Trans. Mosc. Math. Soc.*, **25** (1971), 119–262; **26** (1972), 199–239.
8. C. Carminati, S. Marmi and D. Sauzin, There is only one KAM curve, *Nonlinearity*, **27** (2014), 2035–2062.
9. J. Ecalle and B. Vallet, Correction and linearization of resonant vector fields and diffeomorphisms, *Math. Z.*, **229** (1998), 249–318.
10. L. H. Eliasson, Hamiltonian systems with linear form near an invariant torus, in *Non-linear Dynamics, Bologna, 1988*, pp. 11–29, World Sci. Publ., Teaneck, 1989.
11. L. H. Eliasson, Normal forms for Hamiltonian systems with Poisson commuting integrals-elliptic case, *Comment. Math. Helv.*, **65** (1990), 4–35.
12. L. H. Eliasson, B. Fayad and R. Krikorian, KAM-tori near an analytic elliptic fixed point, *Regul. Chaotic Dyn.*, **18** (2013), 806–836.
13. L. H. Eliasson, B. Fayad and R. Krikorian, Around the stability of KAM tori, *Duke Math. J.*, **164** (2015), 1733–1775.
14. G. Farré and B. Fayad, Instabilities for analytic quasi-periodic invariant tori, https://arxiv.org/pdf/1912.01575.pdf.
15. B. Fayad, Lyapunov unstable elliptic equilibria, https://arxiv.org/pdf/1809.09059.pdf.
16. B. Fayad and R. Krikorian, Some questions around quasi-periodic dynamics, in *Proc. Internat. Congress of Math.–Rio de Janeiro (2018). Vol. III*, pp. 1909–1932, World Sci. Publ., Hackensack, 2018.
17. X. Gong, Existence of divergent Birkhoff normal forms of Hamiltonian functions, *Ill. J. Math.*, **56** (2012), 85–94.
18. X. Gong and L. Stolovitch, Real submanifolds of maximum complex tangent space at a CR singular point, I, *Invent. Math.*, **206** (2016), 293–377.
19. X. Gong and L. Stolovitch, Real submanifolds of maximum complex tangent space at a CR singular point, II, *J. Differ. Geom.*, **112** (2019), 121–198.

20. M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *IHÉS Publ. Math.*, **49** (1979), 5–233.
21. M. R. Herman, *Sur les courbes invariantes par les difféomorphismes de l'anneau*, Astérisque, vol. 1, pp. 103–104, Société Mathématique de France, Paris, 1983. i+221 pp.
22. B. Hunt, T. Sauer and J. Yorke, Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces, *Bull. Am. Math. Soc. (N.S.)*, **27** (1992), 217–238.
23. Yu. S. Ilyashenkoko, Divergence of series that reduce an analytic differential equation to linear normal form at a singular point, *Funkc. Anal. Prilozh.*, **13** (1979), 87–88 (Russian).
24. G. Iooss and E. Lombardi, Polynomial normal forms with exponentially small remainder for analytic vector fields, *J. Differ. Equ.*, **212** (2005), 1–61.
25. H. Ito, Convergence of Birkhoff normal forms for integrable systems, *Comment. Math. Helv.*, **64** (1989), 412–461.
26. A. Jorba and J. Villanueva, On the normal behaviour of partially elliptic lower-dimensional tori of Hamiltonian systems, *Nonlinearity*, **10** (1997), 783–822.
27. V. Kaloshin and A. Sorrentino, On the local Birkhoff conjecture for convex billiards, *Ann. Math. (2)*, **188** (2018), 315–380.
28. T. Kappeler, Y. Kodama and A. Némethi, On the Birkhoff normal form of a completely integrable Hamiltonian system near a fixed point with resonance, *Ann. Sc. Norm. Super. Pisa*, **26** (1998), 623–661.
29. A. N. Kolmogorov, On the persistence of conditionally periodic motions under a small change of the Hamilton function, in G. Casati and J. Ford (eds.) *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, Lect. Notes Phys., vol. 93, pp. 51–56, Springer, Berlin, 1979.
30. J. N. Mather, Differentiability of the minimal average action as a function of the rotation number, *Bol. Soc. Bras. Mat., Nova Ser.*, **21** (1990), 59–70.
31. J. N. Mather and G. Forni, Action minimizing orbits in Hamiltonian systems, in Graffi (ed.) *Transition to Chaos in Classical and Quantum Mechanics*, LNM, vol. 1589, pp. 92–186, Springer, Berlin, 1992.
32. J. K. Moser, On invariant curves of area-preserving mappings of an annulus, *Nachr. Akad. Wiss. Gött. Math.-Phys.*, **1962** (1962), 1–20.
33. J. Moser and S. M. Webster, Normal forms for real surfaces in $\mathbf{C}^2$ near complex tangents and hyperbolic surface transformations, *Acta Math.*, **150** (1983), 255–296.
34. N. N. Nekhoroshev, The behavior of Hamiltonian systems that are close to integrable ones, *Funct. Anal. Appl.*, **5** (1971), 82–83.
35. R. Pérez-Marco, Total convergence or general divergence in small divisors, *Commun. Math. Phys.*, **223** (2001), 451–464.
36. R. Pérez-Marco, Convergence or generic divergence of the Birkhoff normal form, *Ann. Math. (2)*, **157** (2003), 557–574.
37. H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste, Tome I, Chap. 5, Paris*, 1892.
38. J. Pöschel, Integrability of Hamiltonian systems on Cantor sets, *Commun. Pure Appl. Math.*, **35** (1982), 653–696.
39. T. Ransford, *Potential Theory in the Complex Plane*, London Math. Soc. Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995.
40. J. Rauch and M. Taylor, Potential and scattering theory on wildly perturbed domains, *J. Funct. Anal.*, **18** (1975), 27–59.
41. J. Rauch and M. Taylor, Electrostatic screening, *J. Math. Phys.*, **16** (1975), 284–288.
42. H. Rüssmann, Über die Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung, *Math. Ann.*, **169** (1967), 55–72.
43. H. Rüssmann, Kleine Nenner I: Über invariante Kurven differenzierbarer Abbildungen eines Kreisringes, *Nachr. Akad. Wiss. Gött., 2*, **5** (1970), 67–105.
44. H. Rüssmann, On optimal estimates for the solutions of linear partial differential equations of first order with constant coefficients on the torus, in *Dynamical Systems, Theory and Applications*, *Rencontres, Battelle Res. Inst., Seattle, Wash., 1974*, Lecture Notes in Phys., vol. 38, pp. 598–624, Springer, Berlin, 1975.
45. H. Rüssmann, On optimal estimates for the solutions of linear difference equations on the circle, *Celest. Mech.*, **14** (1976), 33–37.
46. K. F. Siburg, Symplectic invariants of elliptic fixed points, *Comment. Math. Helv.*, **75** (2000), 681–700.
47. C. L. Siegel, On the integrals of canonical systems, *Ann. Math.*, **42** (1941), 806–822.
48. C. L. Siegel, Uber die Existenz einer Normalform analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtlösung, *Math. Ann.*, **128** (1954), 144–170.
49. C. L. Siegel and J. Moser, *Lectures on Celestial Mechanics*, vol. 187, Springer, New York, 1971.

50. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.
51. L. Stolovitch, Singular complete integrability, *IHÉS Publ. Math.*, **91** (2001), 133–210.
52. J. Vey, Orbites périodiques d'un système hamiltonien du voisinage d'un point d'équilibre, *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4)*, **5** (1978), 757–787.
53. J. Vey, Sur certains systèmes dynamiques séparables, *Am. J. Math.*, **100** (1978), 591–614.
54. W. Yin, Divergent Birkhoff normal forms of real analytic area preserving maps, *Math. Z.*, **280** (2015), 1005–1014.
55. H. Whitney, Analytic extensions of differentiable functions defined in closed sets, *Trans. Am. Math. Soc.*, **36** (1934), 63–89.
56. N. T. Zung, Convergence versus integrability in Birkhoff normal form, *Ann. Math.*, **161** (2005), 141–156.

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