

The teaching ideas in this book have evolved from the TI-Nspire™ Research Evaluation Project, which provided TI-Nspire software and handhelds to seven English secondary schools and followed the teachers as they began to use the technology with learners. The teachers had support in and away from their classrooms from the wider project team. This book is designed to support you as you begin to use TI-Nspire with learners in your classroom and it begins by describing some of the many mathematics activities and teaching approaches devised during the project by the teachers and the project support team. It continues by suggesting some more advanced and innovative approaches to using TI-Nspire, which require a certain level of confidence with the technology.

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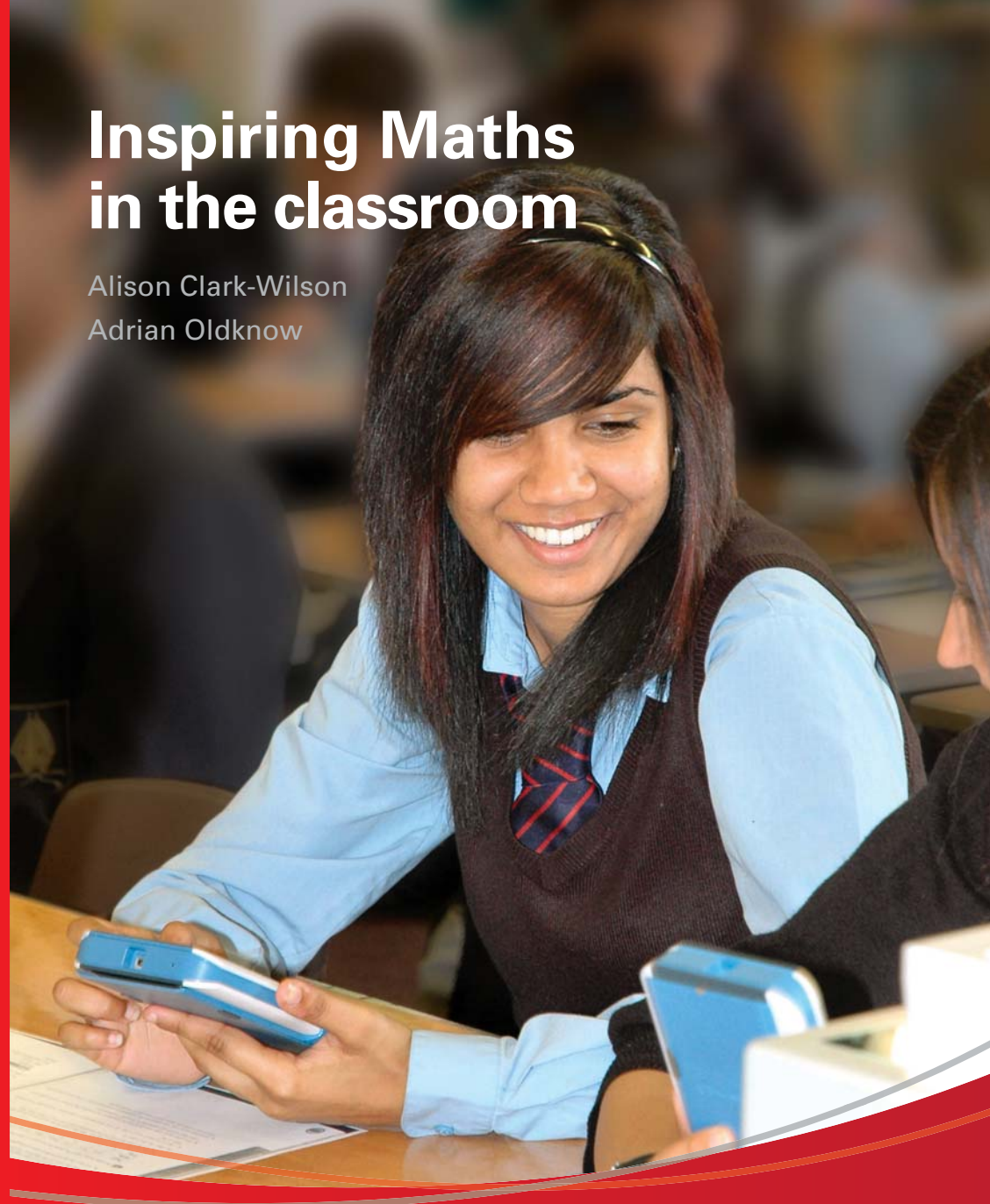
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# Inspiring Maths in the classroom

Alison Clark-Wilson

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University  
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Chichester

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The TI-Nspire™ Pilot Evaluation project took place between September 2007 and April 2008 and involved 15 secondary mathematics teachers from seven state secondary schools in England.

### The teachers and their schools were:

Jay Timotheus, Blue Coat Church of England Comprehensive School, Walsall  
 Margaret Dawson, Blue Coat Church of England Comprehensive School, Walsall  
 Cindy Hunt, Davison Church of England High School for Girls, Worthing  
 Linda Lewendon, Davison Church of England High School for Girls, Worthing  
 Sukaina Sesay, George Green's School, Tower Hamlets, London  
 Frank Opoku, George Green's School, Tower Hamlets, London  
 Pauline Bailey, Ravens Wood School, Bromley, Kent  
 Charlotte Mace, Ravens Wood School, Bromley, Kent  
 Maria Stewart, St Andrew's Church of England High School for Boys, Worthing  
 Rebecca Davey, St Andrew's Church of England High School for Boys, Worthing  
 Debbie Friis, St Paul's Catholic College, Burgess Hill, West Sussex  
 Kevin Cunningham, St Paul's Catholic College, Burgess Hill, West Sussex  
 Katie Smith, Wildern School, Southampton  
 Martin Hole, Wildern School, Southampton (July - Dec 2007)  
 Natasha Cowan, Wildern School, Southampton (Jan – May 2008)

The project was directed by Alison Clark-Wilson of University of Chichester and the project support team was chosen from the existing group of T3™ (Teachers Teaching with Technology™) trainers, an established community funded by Texas Instruments to provide training events and support materials for mathematics teachers implementing handheld technology in their classrooms.

### The TI-Nspire project support team were:

Alan Foster (until November 2007), Educational consultant  
 Barrie Galpin, Educational consultant  
 Nevil Hopley, George Watson's College, Edinburgh  
 Carol Knights, University of Chichester  
 Adrian Oldknow, Emeritus Professor, University of Chichester  
 Chris Olley, King's College, London  
 Jay Timotheus, Blue Coat Church of England Comprehensive School, Walsall

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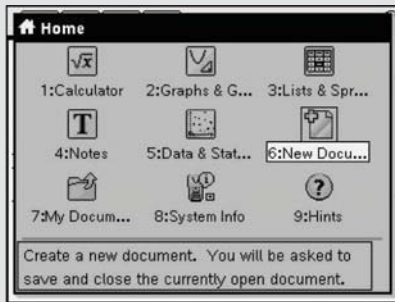
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# Introduction

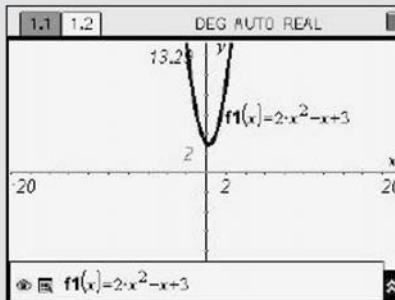
TI-Nspire is an integrated ICT resource which combines a handheld device, a software package and a set of data logging devices to support all aspects of teaching and learning in mathematics and supports practical work in science. The teaching ideas in this book have evolved from the TI-Nspire Research Evaluation Project, which provided TI-Nspire software and handhelds to seven English secondary schools and followed the teachers as they began to use the technology with learners. The teachers had support in and away from their classrooms from the wider project team<sup>1</sup>.

This book is designed to support you as you begin to use TI-Nspire with learners in your classroom and it begins in Section 1 by describing some of the many mathematics activities and teaching approaches devised during the project by the teachers and the project support team. Sections 2 and 3 continue by suggesting some more advanced and innovative approaches to using TI-Nspire, which require a certain level of confidence with the technology.

<sup>1</sup> Clark-Wilson, A. (2008) Evaluating TI-Nspire™ in secondary mathematics classrooms: Research Report, (Chichester, UK: University of Chichester).



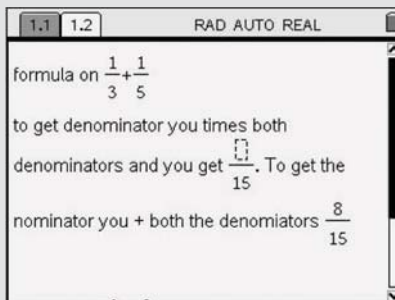
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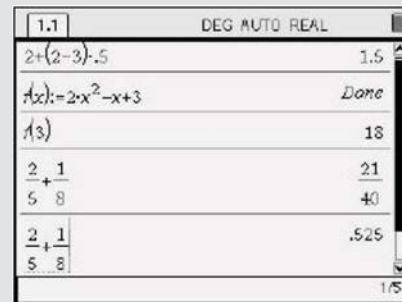
f\_03

	diameter	circumference	
	=capture('d,1)	=capture('c,1)	=b[]/a[]
1	6.62193	20.8034	3.14159
2	6.89638	21.6656	3.14159
3	7.11969	22.3672	3.14159
4	7.25603	22.7955	3.14159
5	7.4793	23.4969	3.14159
C1	=3.1415926535898		

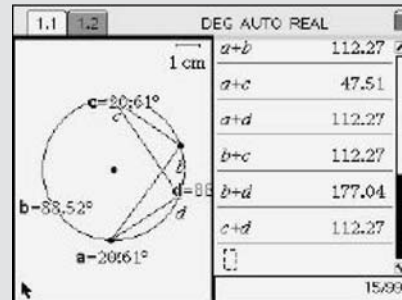
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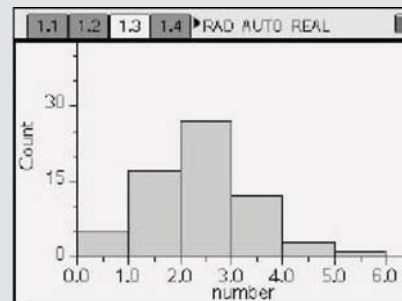
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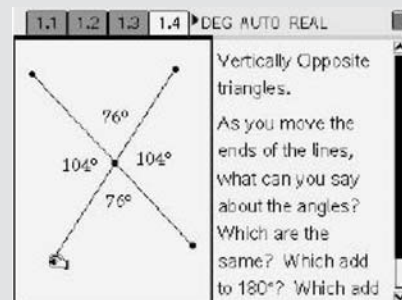
f\_02



f\_04



f\_06



f\_08

## A brief description of TI-Nspire

The individual TI-Nspire applications will feature in detail in this book, so what follows is a brief description of each one. A more detailed technical description of TI-Nspire can be found in the product manuals.

### Home screen

From the home screen you can choose a document to open, or insert one of the available applications into a new or opened document. Changes to the system settings, such as angle format and accuracy can also be made from this screen. f\_01

### Calculator application

Within the Calculator application, it is possible to carry out calculations, define and evaluate functions, display and calculate fraction calculations, define variables and evaluate associated expressions (not shown). If a TI-Nspire file is saved, so are any calculation sequences within the file. f\_02

### Graphs and Geometry application

The Graphs and Geometry application integrates a function graphing environment with a dynamic geometry application. A quadratic function is shown – which was produced by entering its equation on the entry line. Standard linear and quadratic functions can be transformed by dragging, using the NavPad as well as by editing the function. f\_03 It is possible to hide the axes and function entry line by choosing "plane analytic view" if this application is going to be used to construct and manipulate a dynamic geometric figure. If the measurements are defined as variables, it is possible to carry out calculations using these variables on this (and other) pages in the same TI-Nspire file. f\_04

### Lists and Spreadsheet application

The Lists and spreadsheet application uses a conventional spreadsheet format. In addition, the cells in the row marked ♦ are used to define functions applicable to the whole column, which includes a data capture facility from dynamic variables (defined elsewhere in the document). f\_05

### Data and Statistics application

A full range of statistical plots and associated statistical calculations can be produced, using variables defined anywhere in the TI-Nspire file<sup>2</sup>. Statistical graphs can be dynamically explored by dragging individual data points within the data set. f\_06

### Notes application

The Notes application supports text input (with basic formatting) as well as the functionality to write mathematical expressions and draw some standard shapes. It also has a comment facility for the teacher or reviewer. f\_07 Different combinations of any of these applications can be displayed simultaneously, enabling you to design the page using a preset or custom page layout as appropriate to the task. f\_08

<sup>2</sup> Variables are actually defined within subsets of the .tns file, called problems, which prevents global definitions from affecting subsequent activities within the .tns file.

# Section 1

## Some tried and tested teaching approaches at Key Stage 3 and 4

This section of the book is designed to enable you to develop your use of TI-Nspire in Key Stage 3 and 4 classrooms through the professional journeys taken by Annie, Mike and Cathy, each of whom began the project with different levels of confidence and experience with respect to using ICT with their learners<sup>3</sup>.

Annie is the self-confessed ICT sceptic, who uses an interactive whiteboard most days, but this is mainly to organise her lesson resources and support her direct teaching. However she is keen to give the students more access to ICT to think and learn for themselves without having to book an ICT room weeks in advance.

Mike, who has recently qualified, has grown up with technology and thinks nothing of searching the internet for teaching ideas or homework activities for his students. However, he hasn't much experience of using mathematics specific software and is keen to develop this area of his teaching.

Cathy is a self confessed technophile who grasps every new technology as it arrives but also acknowledges that she hasn't necessarily got the staying power needed to support her colleagues to catch up. She is happiest when she is designing creative new approaches to learning mathematical concepts and is always willing to share these with others.

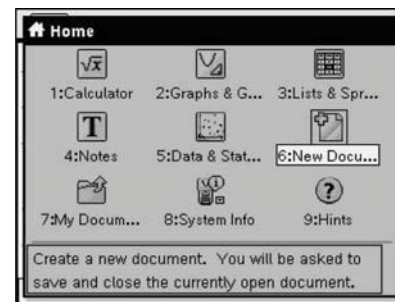
<sup>3</sup> Annie, Mike and Cathy are fictitious names, created to reflect some of the common characteristics of the teachers involved in the project.

# Annie

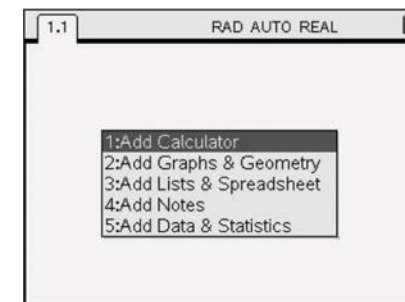
## Getting the handhelds out of their boxes

Annie teaches in an inner city school where the need to provide engaging mathematics tasks to maximise her students' motivation and engagement in lessons is paramount. Consequently, in designing the very first lesson for a middle ability year 9 group she wanted an activity that would allow the students initial curiosities to be satisfied whilst also offering a rich mathematical learning opportunity. Annie did not want to be standing at the front of the class giving lengthy key press instructions for the handheld, so decided on a simple starting point.

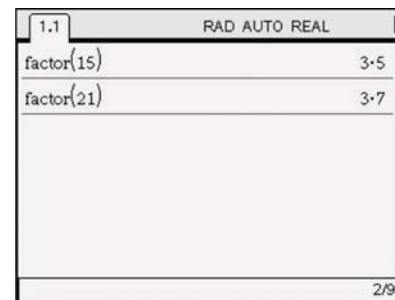
Annie wanted the students to learn about the basics for using the handheld device – how to switch on their TI-Nspire handheld and choose a *New Document* *f\_09* and a Calculator page *f\_10*. She also had set the mathematical aims for the lesson as the consolidation of the students' understanding of prime numbers and the concept of prime factorisation through an exploratory approach. The students would use the built-in command *factor* with a number and see what happened. *f\_11*



*f\_09*



*f\_10*



*f\_11*



The students were supported in the lesson with a recording sheet devised by Annie, which led them through a sequence of questions. They were asked to record what they noticed as they were working through the questions to try and encourage them to generalise and notice patterns. *f\_12*

**An Introduction to TI-nspire**

Press ON

Press Home - The House

The 'donut' is the circle - scroll to highlight number 5 - New Document

Press 'enter' or the centre of the 'donut'

Do you want to save the document ?

Press NO

Scroll to highlight number 1 - 'Calculator'

Press 'enter' or the centre of the 'donut'

We are now ready to begin.

Type the following: Factor(15)

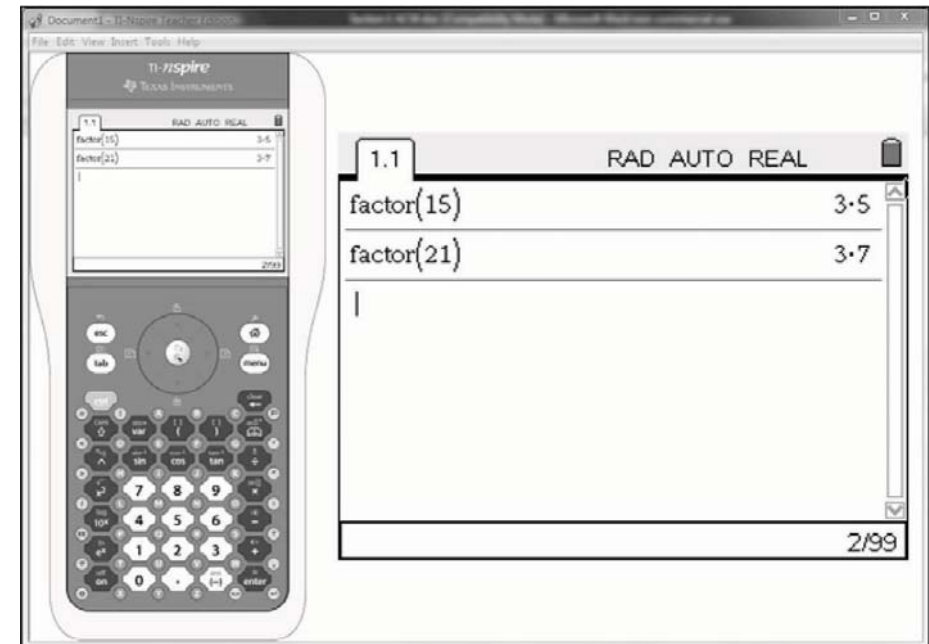
Press 'enter'

What do you see? factor of 15 3 5

*f\_12*

Annie wrote about the students attitudes to the lesson "They were enthusiastic about using the handheld and were sensible about it – Most students worked through the sheet after a little initial guidance. Students were able to work at their own pace and this lesson reinforced previous lessons on lowest common multiples etc." She also added "I was very nervous to use the device even though I am a very experienced teacher of maths. I needed the worksheet for me as well as for them. I was able to refer to the sheet and that helped my confidence. The sheet also allowed pupils to continue with the work whilst I went around to help students with a problem." With the hindsight gained from the lesson Annie also considered how she would redo the worksheet next time, "Next time I would investigate the meaning of factor by getting pupils to find factors of 1, 2, 3, 4 etc. and predict the next."

Annie did not use the TI-Nspire Teacher Edition Software (it was not available at the time) and had not set up the overhead projector panel to project the TI-Nspire handheld screen to the class – she mentioned that this was an oversight on her part and used it for all of the subsequent lessons. The arrival of TI SmartView Emulator within TI-Nspire Teacher Edition Software has meant that a fully functioning TI-Nspire key pad and an enlarged screen make it much easier for the teacher to share the essential first key presses with students at the beginning of the lesson and hold a meaningful plenary at the end of lessons.



*f\_13*

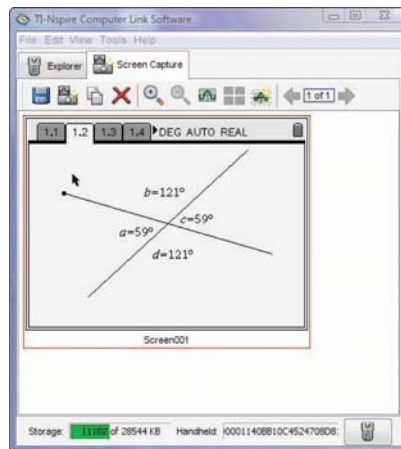
## Moving into geometry – Grab and Drag!

Following the positive experience of her first lesson, Annie decided to be slightly more adventurous and develop a lesson that used the dynamic geometry environment within the Graphs and Geometry application with her Year 8 group. This would require the students to learn how to work in this new application within TI-Nspire, to move between TI-Nspire pages (by pressing **ctrl** and right **→** or left **←**) and also how to use the NavPad to drag objects.

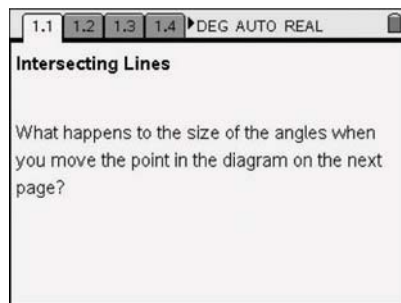
The NavPad, *f\_14* which acts like a mouse, enables the students to grab and drag on-screen objects such as points and lines. As this is similar to most computer game handsets, the students have no difficulties using it!



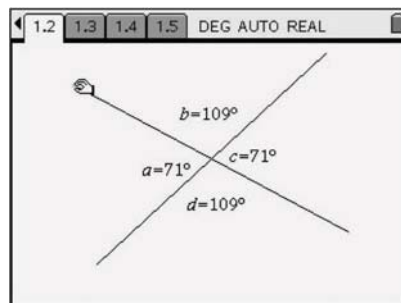
*f\_14*



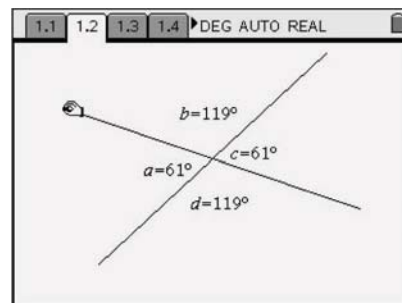
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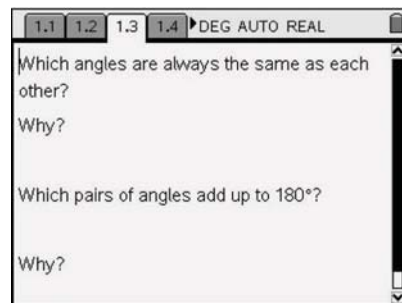
f\_16



f\_18



f\_17







f\_19

Annie decided to adapt an activity that Adrian Oldknow had devised for just this purpose. It also included the use of Notes pages both to outline the task and for the students to add their own text comments. The students would also save their TI-Nspire files onto the handhelds so that Annie could review their work at the end of the lesson.

As this lesson used a pre-designed TI-Nspire file, Annie learned how to transfer this file from her computer to each of the students' TI-Nspire handhelds using the TI-Nspire Computer Link software. As Adrian had emailed her the file, she needed to find where she had saved it on her computer and drag and drop it into the Year 8 folder she had made on the handhelds<sup>4</sup>. Annie also discovered at this stage that she could use the Screen Capture facility to save a copy of the TI-Nspire handheld screen to use in her Smart Notebook file at the beginning of the lesson. f\_15

Having organised the students' access to the TI-Nspire file, Annie was able to think carefully about her aims for the lesson. Apart from developing the student's repertoire of skills with TI-Nspire described previously, she wanted them to explore the relationships between the angles in the diagram and in particular, notice that vertically opposite angles were equal and that the sum of the angles on a straight line is always  $180^\circ$ .

f\_16-19

In order to drag a point the students needed to move the pointer near to it and, when the pointer changed to an open hand , press the  and  keys to close the hand . The NavPad could then be used to move the point and observe the angles change. Annie used the TI-Nspire software on her interactive whiteboard to show the students how to grab and drag the point and also where they should write their responses. A few of the students' initial questions related to the exact manipulation needed to grab and drag the point to vary the angles. However, once they had sorted this out, there were no further problems and Annie concluded that this lesson had given her students a very rich opportunity to analyse the mathematics of the situation by *exploring the effects of varying values and look for invariance*<sup>5</sup>. She also said that "all students were able to see the connections without having to draw and measure" and that she was surprised how, with practice, the students were able to overcome their manipulative difficulties.

<sup>4</sup> Although at the time this needed to be done individually for each handheld, later in the project she was able to send to multiple handhelds using the TI Connect-to-Class™.

<sup>5</sup> One of the process skills in the National Curriculum Key Stage 3 programme of study (2007).

## Linking TI-Nspire applications – getting connected

After a few months of using TI-Nspire, Annie was keen to try to use the facility within TI-Nspire to connect the different representations of a problem using the example of exploring the relationship between the diameter and circumference of circles as described in the back of the book *First steps with TI-Nspire technology for teaching and learning mathematics* that is available for download from the TI website at <http://education.ti.com/sites/UK/downloads/pdf/First%20Steps%20with%20TI-Nspire.pdf>.

Annie chose to work with a group of Year 8 students, who had been using TI-Nspire for a few months and were familiar with both the Geometry, Spreadsheet and Data and statistics pages. This was to be their first exploration of the relationships between different circle measurements and Annie was keen to begin this as she always did, with real cylindrical objects and measuring tapes. The students were seated in groups of four and each group were given three objects of a similar size, one group had a coin, a spice jar and a cotton reel and a mint crisp, whilst another group had two sizes of (plastic) hula hoops and a bin lid.

The students were asked to measure the diameter and circumference of each item and enter the measurements into a Spreadsheet in a new TI-Nspire file. Annie showed them how to label the columns, which would be needed to construct the graph of the data. *f\_20*

Annie then asked the students to add a Data and statistics page and select the diameter to be plotted as the x-coordinates and circumference for the y-coordinates which produces a graph as shown. *f\_21*

The scale for the scatter graph is automatically selected to fit the data in the first instance (although this is easily altered). This meant that each group of students were looking at a different graph –but were there any similarities? At this point, Annie asked the students to make their own conjectures as to how they thought that the measurements of the diameter and the circumference might be related. She also asked the students whether they thought that everyone in the class would have similar conjectures, after all some of the circular objects were very small and others were very large!

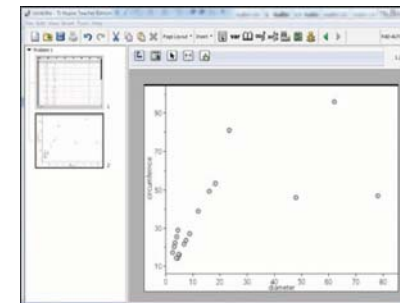
Annie then invited the students to enter their data into a master spreadsheet on her class computer and the class looked at the resulting graph. *f\_22*

On the whole there seemed to be a pattern in the data, however not until a few anomalous points had been sorted out – the students diagnosed and corrected these errors for themselves, which were due to mixing up measurements in centimetres and inches and entering the data in the wrong columns in the spreadsheet. After some rich discussions, the following graph was produced. *f\_23*

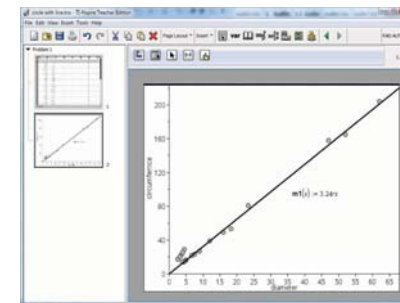
Annie spent some time with the students looking at this graph – she asked questions like “Can you see your three measurements in the graph?” and “What is the graph actually telling us?”

1.1	RAD AUTO REAL		
A	B	C	D
	diameter	circumference	
1	spice	4.2	13.8
2	coin	2.5	7.9
3	cotton	3	9.3
4			
5			

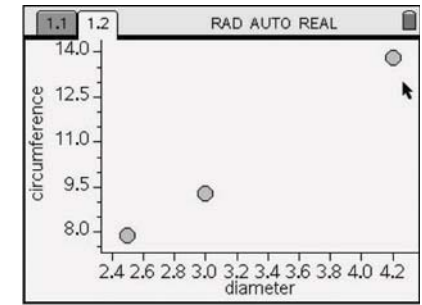
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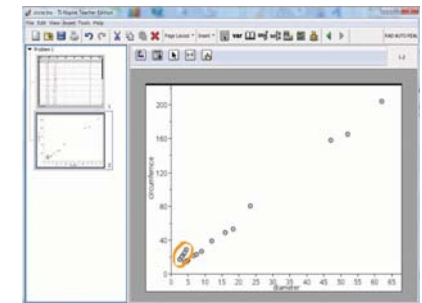
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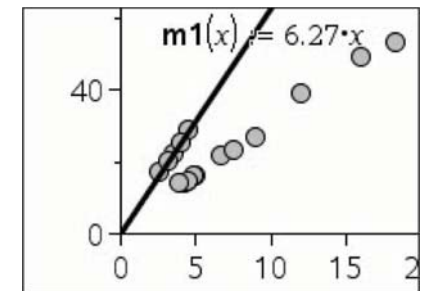
*f\_24*



*f\_21*



*f\_23*



*f\_25*

She asked the students to reconsider their original conjectures – were they still true? Note the interesting set of points that have been circled – we’ll come back to these later!

Annie then added a movable line to the graph, which she and the students agreed ought to be fixed to (0,0). She then rotated the line until the students felt it was in a good position and they discussed what the equation meant. *f\_24*

The class concluded that the circumference was roughly “three and a bit” times the diameter.

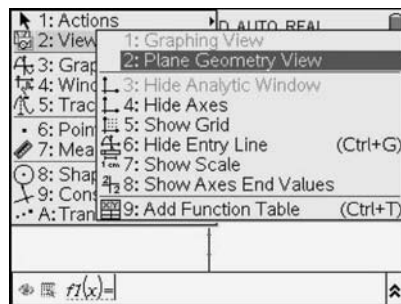
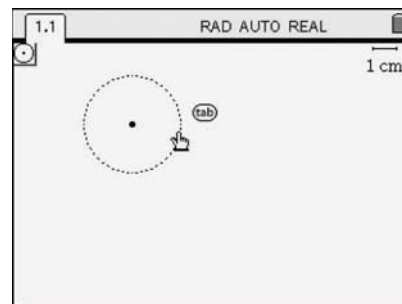


Looking at the data after the lesson, there does seem to be a mysterious cluster of points.  
*f\_25* Further investigation using TI-Nspire reveals that they seem to sit on the line  $y = 6.27x$  – which suggest that the students concerned may have measured the radius instead of the diameter!

The second part of this lesson moved the students from their experimental world of real objects and measurements to the more accurate world of dynamic geometry. Annie wanted to construct a dynamic circle with its diameter and circumference drawn so that the students could graph the accurate data and try to obtain a more accurate value for the ratio of the circumference to the diameter. She asked the students opened a new TI-Nspire document and a Graphs and geometry page.

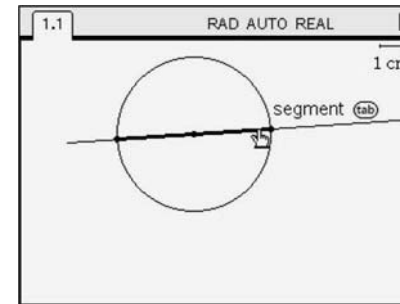
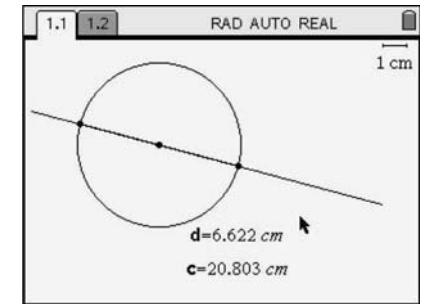
Annie showed the students how to select the Plane Geometry View, which hides the graph axes and the function entry line, which would not be needed *f\_26*

The students were then shown how to select from the (menu) the Shapes option and Circle to enable them to construct a dynamic circle. To achieve this, they needed to press (tab) to fix the centre of the circle on the screen and then use the NavPad to drag the circle to the desired size and then use the (size) key to fix the size of the circle. *f\_27*

*f\_26**f\_27*

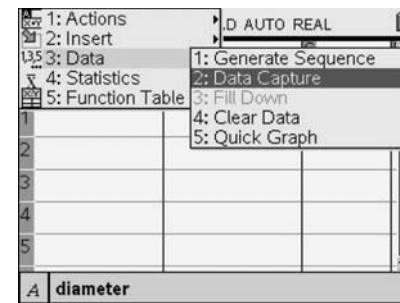
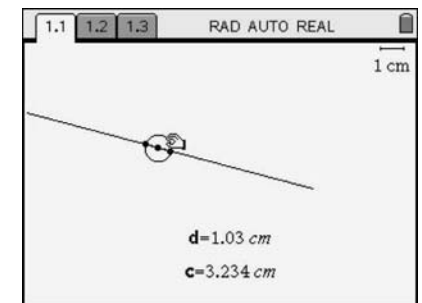
Having drawn the circle, Annie led the students through the steps needed to construct a diameter. This required the students to construct a line through the centre and circumference and find the intersection points. *f\_28*

The diameter can then be drawn as a line segment, which will enable its length to be measured. The length of the circumference was also measured. *f\_29*

*f\_28**f\_29*

An important next step was for Annie to show the students how to “save” the measurements as variables – this would mean they could “capture” the values later on. Annie gave the students the opportunity to select and drag the circumference to enlarge the circle and observe how the values changed. To enable the students to “capture” the measurements, Annie asked the students to add a Spreadsheet page and showed them how to set up the data capture, instructing the handheld to capture the diameter measurements in column A and the circumference measurements in column B. They labelled the columns “diameter” and “circumference” as in the practical lesson. *f\_30*

The students then went back to their dynamic circle construction and, by dragging their circle smaller and larger, generated the measurement data. *f\_31*

*f\_30**f\_31*

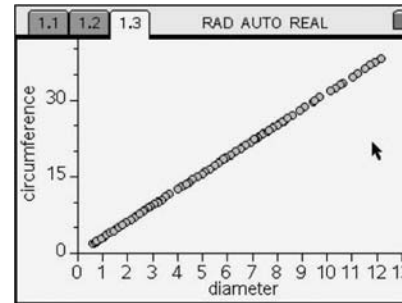
When they moved back to the Spreadsheet table, the table was filled with data. *f\_32*

The next step was to create a scatter plot of the circumferences and diameters.

This seemed to suggest a very direct linear relationship! *f\_33*

1.1	1.2	1.3	RAD AUTO REAL
A	B	C	
diameter	circumference		
=capture('d,1)	=capture('c,1)		
1	6.62193	20.8034	
2	6.89638	21.6656	
3	7.11969	22.3672	
4	7.25603	22.7955	
5	7.4793	23.4969	
A1	=6.6219332524573		

*f\_32*



*f\_33*

Back to the Spreadsheet to calculate the exact value of the multiplier – by defining column C as the circumferences divided by the diameter. *f\_34*

The “multiplier” of 3.1459 (4 d.p.) is obtained and Annie informed the class that this is a number with a special name,  $\pi$ . *f\_35*

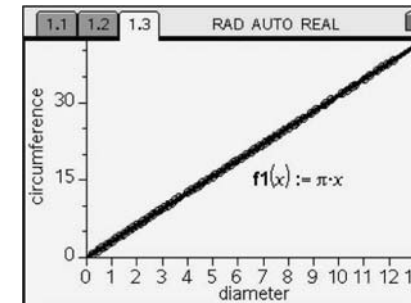
1.1	1.2	1.3	RAD AUTO REAL
A	B	C	
diameter	circumference		
=capture('d,1)	=capture('c,1)	=b/a]	
1	6.62193	20.8034	
2	6.89638	21.6656	
3	7.11969	22.3672	
4	7.25603	22.7955	
5	7.4793	23.4969	
C	=b/a]		

*f\_34*

1.1	1.2	1.3	RAD AUTO REAL
A	B	C	
diameter	circumference		
=capture('d,1)	=capture('c,1)	=b[]/a[	
1	6.62193	20.8034	3.14159
2	6.89638	21.6656	3.14159
3	7.11969	22.3672	3.14159
4	7.25603	22.7955	3.14159
5	7.4793	23.4969	3.14159
C1	=3.1415926535898		

*f\_35*

And to test the rule, Annie entered the equation  $y = \pi x$  which conveniently passed through all of the data points! *f\_36*



*f\_36*

## Where next for Annie?

Annie's confidence to use the technology had developed significantly over the nine months of the project, but she still needed to work things through for herself ahead of the lesson and she always made her own notes with the important key presses and techniques to remember. She liked the idea of saving measurements as variables to enable them to be used elsewhere in the TI-Nspire file, to make calculations, make and test conjectures and draw graphs. Annie was looking forward to developing some activities that used this functionality for the advantage of learners. Having worked through the circle exploration she could see a number of similar geometric examples that would enable a similar approach to be taken.

# Mike

Mike was in his second year of teaching mathematics and, as a mathematics graduate who had always wanted to teach, he was confident about his subject knowledge and had grown up with technology. He even remembered using a graphics calculator at school! Mike knew that the challenge for him was to develop his subject pedagogy, he knew his own mathematics well, but was less confident about how best to teach it. He had had limited opportunities to use ICT in his teaching so far and, although was happily using his interactive whiteboard, he was mainly using his whiteboard software, Microsoft® Office® and a few interactive websites. He was really keen to learn to use some mathematics specific software, and was excited about the possibilities with TI-Nspire.

## Practical starting points – Weighing sweets

This lesson was one that several of the teachers involved in the pilot used, with students from year 7 to year 9, either to introduce the students to the concept of slope in relation to graphs or to consolidate their understanding of the general form for straight line graphs, normally described as  $y = mx + c$ . This lesson was created using an idea by Annette Ziedler of Malmo University, Sweden who is a member of an International group of researchers interested in developing rich classroom assessment tasks that use technology.

In the task the students are given an opened pot of small sweets, some digital scales and a TI-Nspire handheld and are asked to generate their own interesting mathematical questions. The class are encouraged to share their questions and, supported by the teacher, arrive at a decision about which questions would be appropriate to try to answer. They are then asked to devise their own “experiment” to accomplish this.



f\_37

In Mike's class, the students were highly engaged in the initial activity and came up with questions like:

- |                                    |                                    |
|------------------------------------|------------------------------------|
| How much does one sweet weigh?     | How much do all the sweets weigh?  |
| How much do half the sweets weigh? | Do they all weigh the same amount? |
| How much does the container weigh? |                                    |

And the occasional obvious question such as: Are the sweets tasty?

Mike agreed some ground rules with the class...



## Sweeties!

### Rules



1. You cannot weigh one sweet individually
2. You cannot weigh the empty container
3. You cannot count the number of sweets in the pot

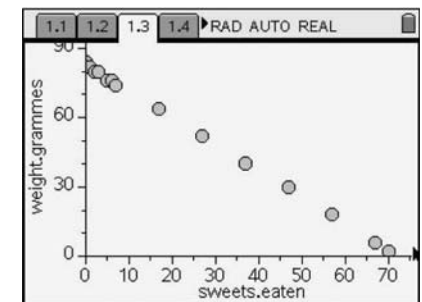
f\_38

The students were shown how to input data into the Spreadsheet page and very quickly realised that by consuming the sweets they could gather some data to help them to solve their problems! f\_39

And by producing a graph, the students could begin to hypothesise about how this might help them to answer their questions. f\_40

1.1	1.2	1.3	1.4	PAD AUTO REAL
A	sweets.eaten	B	weight.grammes	
1	0		84	
2	1.		82.	
3	2.		80.	
4	3.		80.	
5	5.		76.	
B1	84			

f\_39

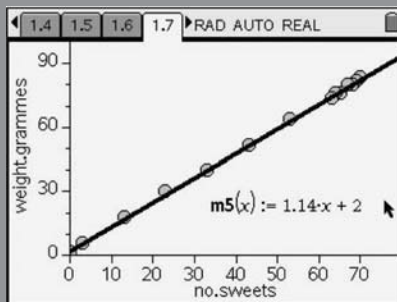


f\_40

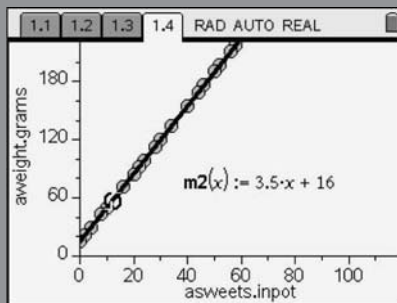
	sweets.eaten	weight.gram...	no.swe...
			$=70-a[]$
1	0	84	70
2	1	82	69
3	2	80	68
4	3	80	67
5	5	76	65

C no.sweets:=70-a[1]

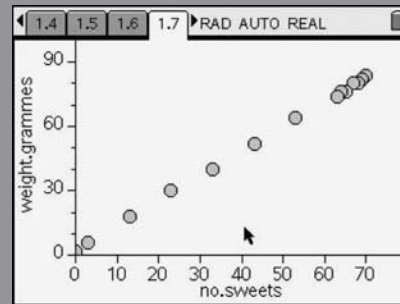
f\_41



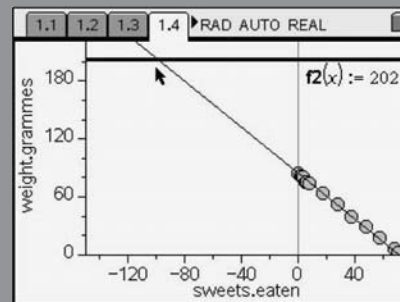
f\_43



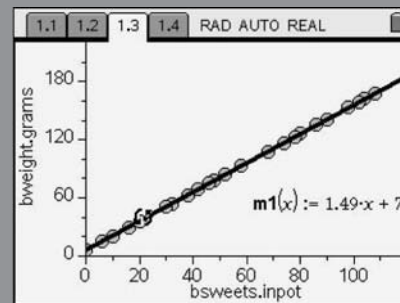
f\_45



f\_42



f\_44



f\_46

Having eaten all of the sweets the teacher discussed the idea of “inverting” the graph by imagining that the container was empty and that the students were adding the sweets back into it again and what might happen to the total weight of the sweets and container.

So, using the data shown previously, 70 sweets and the container would weigh 84g, 69 sweets and the container would weigh 82g and so on...

By entering a formula ( $=70-a$ ) into column C, this data is quickly generated in the Spreadsheet. f\_41

and the resulting graph gives an interpretation of data. f\_42

By adding a movable line, a “rule” is generated which has some meaning for the students, the steepness relating to the weight of one sweet and the intercept the weight of the container. As it is movable, the students rotated and slid it so that it “fitted” over their data points. f\_43

The important discussion then arose as the students discussed what the 1.14 represented and what the 2 represented in relation to their original problem. Could they use this rule to answer all or some of their questions?

An alternative strategy might have been to have extrapolated the line produced by the original line backwards to see where it would cross the line  $y = 202$ , they would be able to work out how many sweets were originally in the container if it weighed 202g. f\_44


In Mike’s lesson, during the plenary, he was keen to see how much the students had understood in relation to the algebraic rule that TI-Nspire had generated from the movable line that the students had dragged into position. He showed the following two graphs to the students and asked them to discuss what was similar and what was different about the two graphs, what sweets they thought the graphs might represent and what sort of containers might have been use.


Although, prior to this lesson the students had not yet met linear functions in the form  $y = mx + c$ , Mike was confident that, by introducing the concept of straight line graphs through a practical task, the students had been able to interpret the meaning of the algebra within the context of their problems. f\_45-46





## New approaches into circle theorem

Mike was keen to start to use the dynamic geometry system in TI-Nspire which is accessed from the Graphs and Geometry application. Previously he had played with Geometer's Sketchpad and used it for a few class demonstrations but had not yet developed any lessons where the students worked with it for themselves. He decided to develop an activity where the students would construct some simple figures for themselves, measure some angles and make and test some conjectures.


Mike wanted the students to "discover two circle theorems – opposite angles in a cyclic quadrilateral make 180 and angles in the same segment are equal" so he guided the students, using the TI-Nspire software displayed on his whiteboard using the SmartView display option. This enabled both Mike and his students to use the TI-Nspire handheld displayed on his interactive whiteboard which worked in the same way as the handheld. Before they started, Mike asked the students to select the  option and change the System settings for their handheld to "Float 2" (displaying 2 decimal places without actually rounding the numbers) and the Angle measurement to degrees.


Mike asked the students to create a new TI-Nspire document and insert a Graphs and geometry page. They then selected  and 2:View to choose 2:Plane Geometry View.



This is a quick way to hide the axes and the Entry line. *f\_47*

They then selected   to find the 5:Page layout menu from where they could split the screen horizontally and add a Calculator page to the right hand pane. *f\_48*

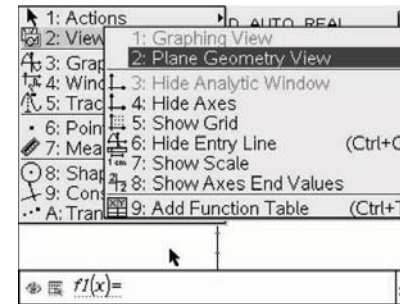
In the left hand pane Mike showed the students how to construct a circle and four segments that intersected on the circumference to create an inscribed quadrilateral. At this stage he asked the students to use the NavPad to drag the vertices around the circumference to ensure that the quadrilateral didn't fall apart. Some students had to reconstruct their segments if this happened!

Mike then showed the students how to label the vertices by selecting a point followed by  and 1:Actions and choosing 6:Text.

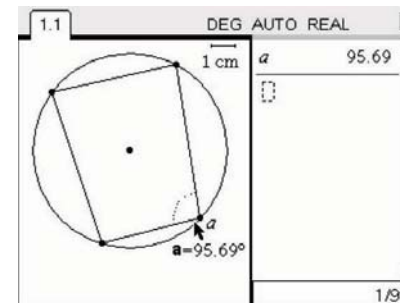
He then talked the students through the steps to measure an angle. Firstly they chose this option by selecting  7:Measurement and 4:Angle. Secondly they identified the angle to be measured by selecting the three vertices that defined the angle one after another. *f\_49*

Once the angle measurement was on the screen, Mike asked them to select it and, by pressing  followed by the  key, a letter could be assigned to this measurement. (Typing this letter in the adjacent Calculator window would reveal its instantaneous measurement.)

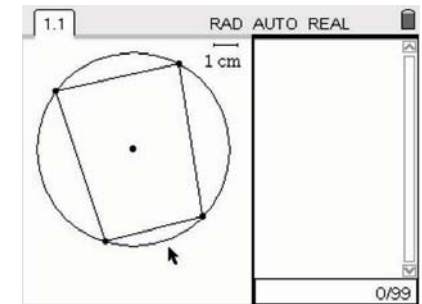
The students repeated this for all four of the angles and then, by dragging the vertices around and observing different situations, began to conjecture what was happening. They were able to test any conjectures in the calculator window. *f\_50*



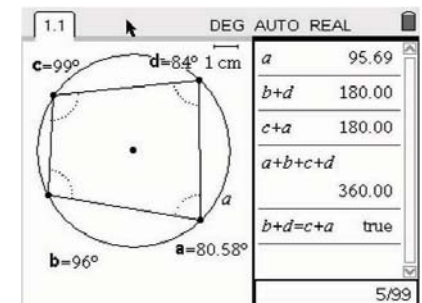
*f\_47*



*f\_49*



*f\_48*



*f\_50*

Mike was happy with the outcomes of this lesson. He said "I feel that they are stronger at applying these theorems as they have been able to discover them for themselves and play about with the circle and cyclic quad as much as they liked and still see that they hold... I feel that my teaching of this topic has been enhanced by this activity. Although it could have been done on a geometry program in a computer room, the use of the TI-Nspire was less disruptive and, as this class have used them before, it was easier for them to access. I feel that the pupils will remember these circle theorems as they have had a practical lesson to discover them rather than being shown or told them and then simply applying them." Mike's students were equally positive about their learning and one group agreed that it was beneficial to have been able to generate lots of examples quickly.



## Using TI-Nspire to introduce trigonometric ratios

Following on from his success with the Circle theorem lesson, Mike had been inspired by the activity to think about how he could set up a similar style of lesson to enable the same class of Year 9 students to find out about the important relationships between the side lengths of right angled triangles. Essentially, he wanted to introduce his students to the trigonometric ratios, but wanted them to discover them in the hope that this would be a more memorable way of learning for them.

Mike wanted to set up a group activity whereby each group would gather data relating to a given dynamic right angled triangle but, unknown to them, each group of students would be exploring a different situation.

He enlisted his project mentor's help in designing this lesson as he knew the construction that he wanted to produce but wasn't quite sure how to do it! Carol Knights worked with Mike to develop a dynamic right angled triangle that had been defined by an angle measurement that would later be hidden from the students, but would enable one file to be "tweaked" to produce the six different scenarios.

This is what they came up with...

By starting with a line segment defined by AB and adding the "text"  $60^\circ$  to the screen, this angle can be used as an angle of rotation to rotate the segment about one of its ends (in this case the point A) to produce a new segment AA'. By constructing the perpendicular bisector of AA', the point C can be constructed. By adding the segments through the three vertices, A, B and C, the construction lines can then be hidden leaving just the right angled triangle and a mysterious number on the screen. As this number represents an important variable in this exploration, by editing it, Mike and Carol were able to efficiently produce the different starting points for each group. They had also measured and saved the lengths of the three sides as A, B and C and added a Calculator window to the right hand side of each page.

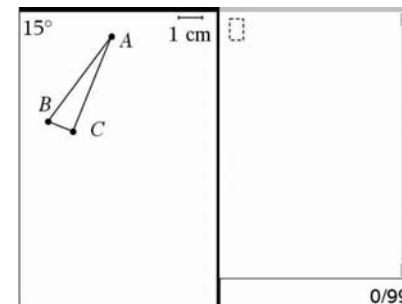
Mike and Carol decided to set this lesson up as a rainbow activity in which each group would all have a handheld each with the same preloaded file. Each group would then explore how the calculated ratios of the measurements of pairs of sides changed as they dragged the only movable point on the shape. (The students had to find out which one this was). They would then use the Calculator window to conjecture and test their findings and record these on a Notes page in the document. The group would then discuss their findings such that each person in the group could explain them to someone else and then the whole class was rearranged so that they were in a new group consisting of one person from each of the original groups. *f\_51-52*

Mike and Carol decided to use coloured stickers on the handhelds to assist this process! The students were quickly engaged with the task and found different ways of recording the values that they were calculating. The task instructions were included in the file. *f\_53-54*

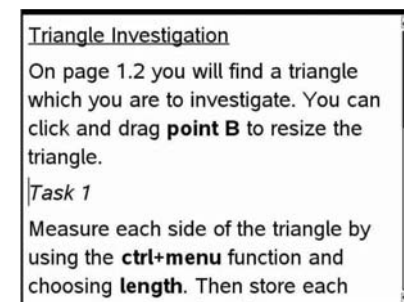
Mike then held a lesson plenary where he asked each student to "notice" the number on the screen and make suggestions as to what it might represent. By displaying the different initial files the students were able to make the connection to the angle between side AB and side BC. He then asked the students to feedback the ratios they had discovered to produce a summary table of angles and the related ratios. After the lesson, Mike commented about how the students expected to find patterns within the lists of ratios they calculated. Some students thought they had found a pattern and so generated other "results" based on the pattern they had perceived... this was noticed by the class when their results did not concur with everyone else's.

## Where next for Mike?

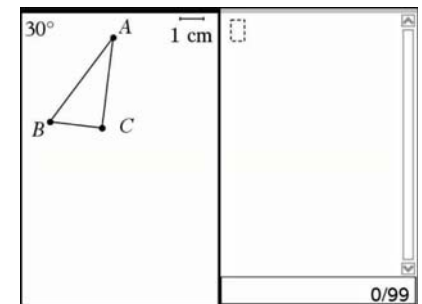
Mike was very positive about the new pedagogical skills he had developed and the way in which they had allowed the students to be much more involved in their learning. His able year 9 group had responded well to the extra responsibility they had been given and they clearly enjoyed the group work. Although these lessons had taken more preparation, Mike saw this as an investment as he could adapt the approach for different levels of students. For example, he could see the rainbow groups as being a strategy he could use when students were investigating other mathematical situations where there were several variables to explore. An example of this might be investigating the area of isosceles triangles where both the base and the height could be varied.



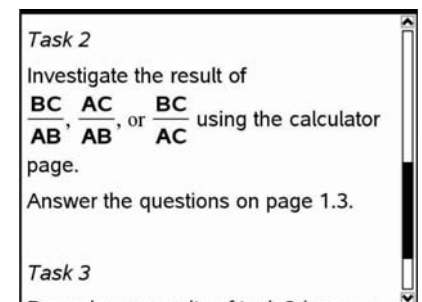
*f\_51*



*f\_53*



*f\_52*



*f\_54*

# Cathy

Cathy is an advanced skills teacher of mathematics with a particular interest in using technology and was introduced to a range of resources (Coypu, Cabri-geometre and graphics calculators) whilst on her initial teachers education course. Over the years she had developed many ICT activities for mathematics schemes of work years and she generally embraces new technologies, always looking to see how she can adapt old approaches and develop new ones. She sees ICT as providing an ideal opportunity for the students to explore the mathematical ideas for themselves and prefers them to have hands-on personal access rather than only watch demonstrations in whole class sessions. Consequently, although she has worked in a number of schools with varying access to computer suites, she has always used graphical calculators, which are a resource that the mathematics department can manage for itself.

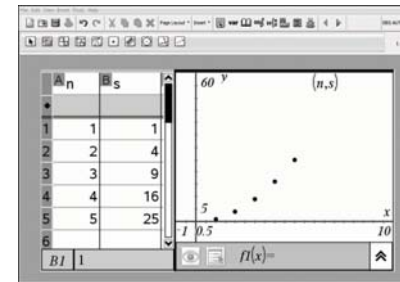
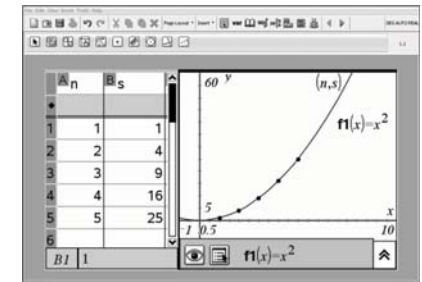
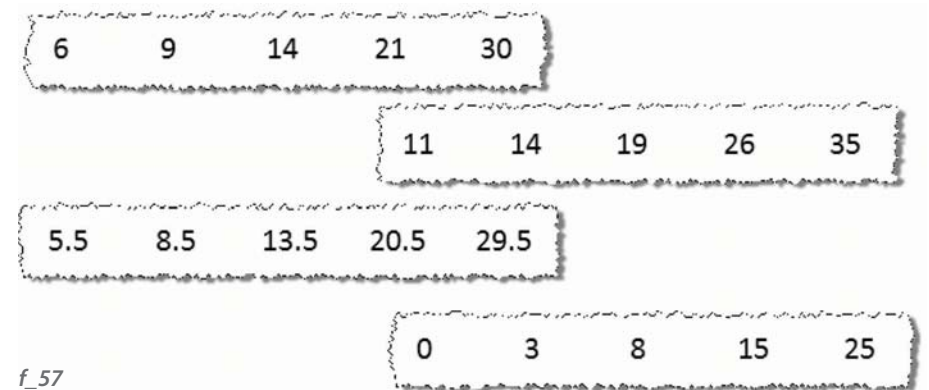
## Familiar territory – introducing quadratic functions

In her early TI-Nspire lessons, Cathy began by taking an approach she had used many times using graphing calculators in which students explore a physical growing pattern and attempt to model it mathematically by entering equations to “fit the data”.

She started by displaying the pattern of square numbers on the board, 1, 4, 9, 16 and asked her year 8 middle ability class what they noticed about the pattern – How was it growing? Could they predict the next number? and so on. She then displayed a page from a TI-Nspire file that she had prepared before the lesson (and also pre-loaded onto the class set of handhelds). She had chosen a split page with a Spreadsheet on one side and a Graphs and geometry page on the other. Cathy had set up a Scatter plot so that, as the data was entered into the columns defined as “n” and “s”, so the coordinate points were plotted on the graph. *f\_55*

Cathy then asked the students to suggest different “rules” (in terms of x and y) to enter as a function to try to fit through the points. Initially the students made suggestion like  $f(x) = 1$  and  $f(x) = 4$ , which produced horizontal lines which the class discussed. After a few minutes to explore this for themselves, the class arrived at  $f(x) = x^2$  which seemed to make sense. *f\_56*

Cathy then gave the students a set of cards, each with a different number sequences. Of course Cathy had chosen these because they were of the type  $y = x^2 \pm a$ , but this was not obvious to the students! *f\_57*

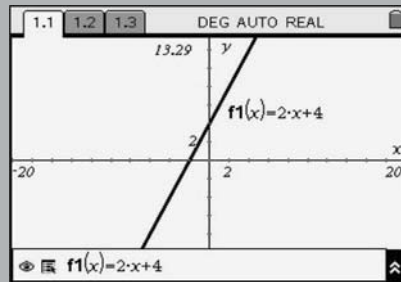
*f\_55**f\_56**f\_57*

The students were then asked to choose a sequence and, using their TI-Nspire handheld, open the file that Cathy had preloaded and input the data into the Spreadsheet and try to work out the equation that would fit exactly through the coordinate points that had appeared on the graph.

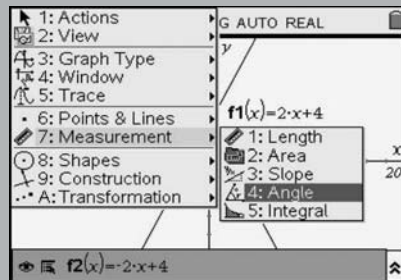
Cathy had given the students a set of card squares and she asked the students to think about how the original square number pattern has been transformed to produce the new sequence.

This is probably easiest to see with the sequence 0, 3, 8, 15, 24... where each “square number” has been reduced by one.

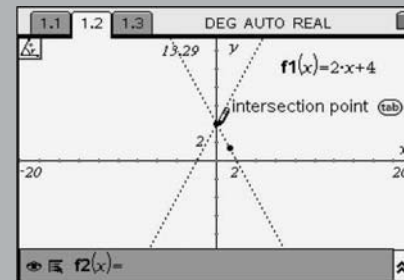
Finally, the students were asked to produce a large poster that displayed their physical pattern of squares, their sequence of numbers, a screen shot of their graph and, most importantly their explanation of why their rule fitted their sequence. Cathy concluded that the use of TI-Nspire has supported the students to make important connections between the different representations of the situation in addition to being the first time they had met curved graphs!



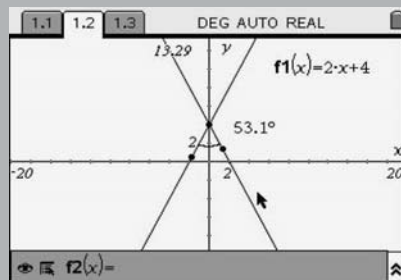
f\_58



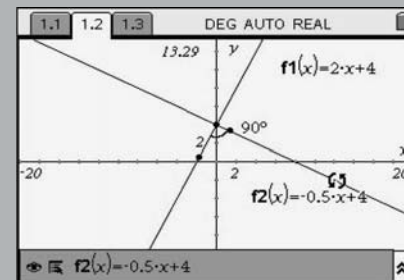
f\_59



f\_60



f\_61



f\_62

1.1	1.2	1.3	1.4	DEG AUTO REAL
A	slope1	B	slope2	
				$\neq a \cdot b$
1	2	-0.5		
2	3	-1/3		
3	4	-1/4		
4				
5				
C	$\neq a \cdot b$			

f\_64

## Grabbing and dragging functions – combining approaches from algebra and geometry

Once Cathy had seen that she could combine ideas and approaches from algebra and geometry on a Graphs and Geometry page, she came up with an innovative new approach to allow her students to explore the relationship between the gradients of linear functions where the lines intersect perpendicularly. This used an innovative feature of TI-Nspire – the ability to create a function from its equation and then drag it (by rotating or translating) and observe how its equation changes. Cathy had previously taught some work on gradients and intercepts of straight line graphs to this class and, as she wasn't sure how much the students would recall from this work, she decided to develop an activity which would allow students to revisit this topic whilst also providing them with an opportunity to explore the relationship between the gradients of lines which are perpendicular to each other.

Cathy started with a blank Graphs and Geometry page and asked her students to generate a linear function by entering in its equation on the entry line at the bottom of the screen as shown. The students had to learn to use the (tab) key to move between the Graph screen and the entry line and were not put off by the function notation syntax  $f1(x)$ . f\_58

The students were asked to guess the function of a second line that would cross their initial line at right angles and enter this in a similar way to before as  $f2(x)$ . f\_59

Cathy then showed the students how to check how accurate they were by measuring the angle between their two lines using the Measurement tool accessed from the (menu) key. f\_60

The students were shown how to measure the angle by pressing the (2) key and selecting a point on the first line, followed by the intersection point of the two lines and a point on the second line. f\_61

The angle then appears on the screen. f\_62

They were then shown how to move their pointer to one end of this new line and, when the pointer changed to ↻, grab the line by pressing (ctrl) and (2) and use the NavPad to drag the line until it was as near to 90° as was possible.

Finally Cathy asked the students to “notice” the equation of the second line. f\_63

Some students inserted a Spreadsheet page and entered the values of the slopes of the pairs of lines they had explored into two lists. They could then explore the relationship between the numbers. f\_64

After the lesson Cathy commented particularly about the way that her students had worked during this lesson – all of the students drew various graphs on the handhelds and discussed their findings with each other, most were able to verbalise their findings and some were able then able to write down rules and explanations with little support (all pupils did this with some support). Cathy also said that her students really appreciated the ability to move the line on the screen and then be told the equation of the new line. What is particularly interesting about this lesson approach is the way in which the technology enabled to the students to connect the algebraic and geometric ideas to support them to generalise from what they were experiencing and observing. The dynamic nature of the task also meant that many more cases could be examined and this supported the students to make and test their conjectures in a mathematically engaging way.

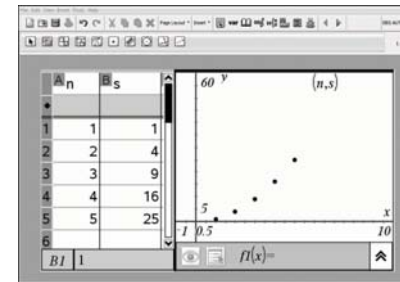
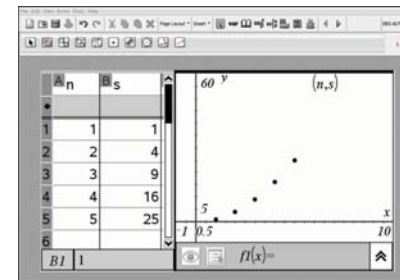
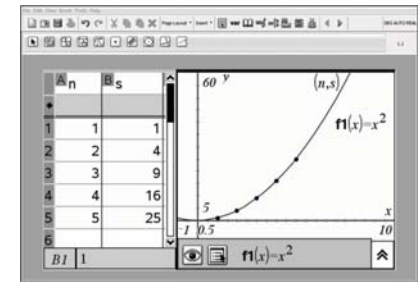
### Exploiting multiple representations - solving quadratics

When it came to teaching a trial and improvement approach to the solving of quadratic equations to her year 11 group, Cathy has previously taught this by encouraging the students to record the results in a table whilst they substituted values of  $x$  into any given function. However, she often felt that difficulties with substitution (particularly negative numbers) and following an instrumental approach meant that the students did not seem to make the connections between what they were doing and what this meant graphically. She decided to use the multiple representations within TI-Nspire to try to bring these two experiences together.

Initially she introduced her students to function notation within a new Calculator page. They defined a quadratic function and substituted some values to evaluate the function and make sense of what TI-Nspire was doing. They continued to substitute decimal and fractional values for  $x$  and discussed what the “answers” meant. Cathy spent time discussing with students the concept that the “solution” of an equation was the value of  $x$  when the equation was equal to zero. *f\_65*

Cathy showed the students how to set up the spreadsheet such that, for any value typed into column A, the equivalent value of the function would be automatically calculated in column B. (To do this, the students typed  $\ominus \text{ (2) (A) } x^2 \text{ (5) (A) } \text{ (6) }$  into the cell shown opposite. The students defined column A as  $x$  and column B as  $y$  which allows the variables  $x$  and  $y$  to be used anywhere in the TI-Nspire file. *f\_66*

In the final stage, Cathy asked the students to add a new Graphs and Geometry page and, by pressing  $\text{ctrl} \text{ (menu)}$ , selecting a Scatter plot. Pressing  $\text{ctrl} \text{ (menu)}$  enabled the students to assign the variable  $x$  to be plotted on the  $x$ -axis and similarly the variable  $y$  to be plotted on the  $y$ -axis. This produced a graph of all the data the students had generated in the spreadsheet. As new data was generated, so the graph would continue to be updated. *f\_67*

*f\_65**f\_67**f\_66*

After the lesson, Cathy concluded that she would use this approach again as it seemed to really link the idea of the graphical solution with the process of trial and improvement – something that she had not been able to do easily previously. She could also see a way of bringing in the factorised form for a quadratic as it would be possible to superimpose functions of the form  $f(x)=(x-x_1)(x-x_2)$  where  $x_1$  and  $x_2$  are the solutions found from the trial and improvement strategy.

### Where next for Cathy?

Undoubtedly, Cathy's experience of a range of different software packages has enabled her to connect the different applications within TI-Nspire more quickly, as she has a good working knowledge of the different applications that it uses. She is yet to use TI-Nspire connected to a motion detector (Calculator Based Ranger or CBR) or a temperature probe, both of which she had previously used with a TI-84 graphics calculator displayed to the whole class using a ViewScreen and an overhead projector. She had had many successful lessons with students “walking” distance-time graphs and modelling cooling curves and is keen to learn how to adapt these lessons with TI-Nspire.

Cathy is also keen to exploit the idea of linking variables between different applications, particularly within the geometry screen to enable students to conjecture ideas and then test them algebraically, as Mike did in his “Circle theorems” lesson.



# Section 2

## Where next – innovative ideas for the classroom

In many ways TI-Nspire is the mathematical equivalent of the set of “Office” tools – and so it is a considerable challenge to identify just a few key techniques which are of general use in solving problems and modelling realistic situations. The great thing about the software is that while it is designed for use in an educational context, it is equally accessible to learners, teachers and to problem solvers.

One extremely important use of ICT in mathematics is for **optimization** – finding a best (or at least a good) solution to a problem requiring some function to be maximised (e.g. profit) or minimised (e.g. time). In order to illustrate the wide range of approaches possible we will use a simple problem about maximising the area of an isosceles triangle.

A Northumberland sheep farmer has two pieces of fencing, each 10m long which she wants to fix to a straight stone wall to make a triangular sheep pen whose area is as big as possible. We can make a “scale model” by using, say, 1cm to represent 1m.

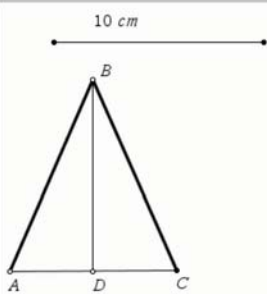
We will look at a variety of different ways we can approach such a problem using TI-Nspire: e.g. numeric, graphic, symbolic, geometric, statistic, algorithmic and synoptic.

ABC is an isosceles triangle with  $AB = BC = 10$  cm. Find the area of the triangle if:

a)  $\angle BAC = 30^\circ$   
b)  $BD = 6$  cm  
c)  $AD = 5$  cm.

Choose one of the variable measurements to be your independent variable  $x$ . What are the minimum and maximum values the variable  $x$  can take?

Can you sketch a graph of what you think the area will be as a function  $y(x)$  of  $x$ ?



## Numeric

For the numeric approach we will illustrate the uses of both Calculator and Lists & Spreadsheet pages, as well as storing values in variables.

In many cases it is more helpful to use the Lists & Spreadsheet page as this will cause values to be recomputed when a value or formula is changed. We define the column A to hold the variable  $ht$  as a list defined to be the sequence of numbers  $n$  starting from  $n=1$  and going on for 9 terms. **f\_69**

If we make a little change in the definition of the variable in column A we can make it much easier to “zoom in” on any particular value which seems interesting. **f\_70**

$h:=5$	5
$h \cdot \sqrt{100-h^2}$	43.3013
$h=\{1,2,3,4,5,6,7,8,9\}$	$\{1,2,3,4,5,6,7,8,9\}$
$h \cdot \sqrt{100-h^2}$	$\{9.94987, 19.5959, 28.6182, 36.6606, 43.3013, 48.49.99, 48.39.2301\}$

Using a Calculation page we can store a value in  $h$  to represent the height  $BD$  say, and compute the corresponding value of the area  $a$ . A variable can also hold a list of elements, so we can see that the largest area for the integers 1 to 9 is  $49.99 \text{ m}^2$  when  $h = 7$ .

f\_69

$ht$	$a$
$=\text{seqn}(n,1,9)$	$=ht \cdot \sqrt{100-ht^2}$
1	9.94987
2	19.5959
3	28.6182
4	36.6606
5	43.3013
6	48
7	49.99
8	48
9	39.2301

Using a Lists and Spreadsheet page we can define the A column to hold the values of the variable  $ht$  defined as a sequence of values of  $n$  starting from 1 going up to 9. The column B holds the area  $a$ , given by a formula in terms of the height  $ht$ . We can now edit the formula for  $ht$  to “zoom in” around  $ht = 7$ .

f\_70

Here the variable  $hz$  is defined by the recurrence formula:

$$u(n) = u(n-1) + 0.01 \text{ with } u(1) = 7 \text{ and } n=2,3,4,\dots,10$$

The column B now holds the list of values of the variable  $az$  defined in terms of  $hz$ . **f\_71**

$hz$	$az$
$=\text{seqn}(u(n-1)+0.01,7,10)$	$=hz \cdot \sqrt{100-hz^2}$
1	49.99
2	49.9926
3	49.9948
4	49.9966
5	49.9981
6	49.9991
7	49.9998
8	50
9	49.9998
10	49.9993

Here the columns have been given new variable names  $hz$  and  $az$  – and the formula for  $hz$  can be easily edited to zoom in numerically around any value. So we can see that the maximum value for the area appears to be  $50 \text{ m}^2$  when the height of the triangle is about 7.07 m.

f\_71



We can generate a scatterplot from the numerical data in any pair of lists of the same length – which can be generated in a Calculator or Lists & Spreadsheet - but we'll need rather a lot more data points than just 10:

$hz := \text{seqn}(u(n-1)+0.1, [0], 100)$  as the horizontal variable and the list  $az := hz * \sqrt{(100-hz^2)}$  as the vertical variable. Within that page there is an Analyze icon which brings down a menu including Plot Function – so we can draw the graph of  $f_1(x) = x * \sqrt{(100-x^2)}$  and show, of course, that it goes through all the data points. So that leads us into the main way of graphing i.e. using a Graphs & Geometry page. In the Data & Statistics page we already defined the function  $f_1(x)$  so it's there waiting to be drawn. If you start a New Problem then all the earlier definitions are “forgotten” and you can enter your function as a new definition for  $f_1(x)$  if you prefer. You can use Windows Setting to control the scales on each axis, and with a right click on the graph you can define its Attributes, and choose the layout for the equation. Using Graph Trace you drag a point along the curve, and when it reaches a maximum (or minimum) it shows an “M” symbol, allowing you to press Enter and capture the point. **f 73**

(Typical location of the pdf is **C:\Program Files\TI Education\TI-Nspire\help\TI-NspireHelp\_EN.pdf**)



1. `nfMax(Expr, Var) ⇒ value`
2. `nfMax(Expr, Var, lowBound) ⇒ value`
3. `nfMax(Expr, Var, lowBound, upBound) ⇒ value`

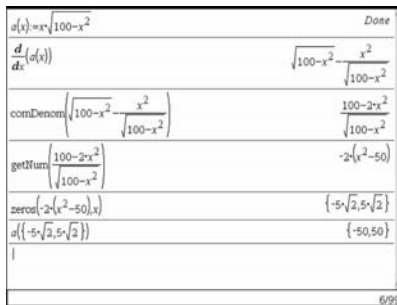
In either case the Var in question is called  $x$ . But our function is not a very well behaved one! It is not defined for  $x > 10$  or  $x < -10$ . There is a minimum at around  $(-7.07, 50)$ , an inflection at  $(0, 0)$  and a maximum at around  $(7.07, 50)$  – but the behaviour at  $(-10, 0)$  and  $(10, 0)$  is distinctly nasty! (Of course negative values of  $x$  make no sense for the actual triangle problem we started with – but we haven't told the software that we don't want to consider negative values of  $x$ . As you see from the screen below, neither of the first or second forms comes up with a solution, whereas the third one does!

So now we have a nice way of showing at the same time a numeric representation in the form of the table, together with a graphic representation in the form of the curve and a symbolic representation in the form of the algebraic expression for the function  $f_1(x)$  whose graph and table are displayed. *f\_75*



## Symbolic (optional: CAS)

TI-Nspire is available in two versions known as CAS and non-CAS. “CAS” stands for “Computer Algebra System” and implies that the software has algorithms for symbolic manipulations in algebra, including differential and integral calculus. So here is a little glimpse into the CAS world to show just how powerful such tools are. The second line shows a symbolic derivative, and the third shows it manipulated algebraically to a proper fraction. So we can see that the denominator can be zero (at  $x = -10$  or  $x = 10$ ) confirming the infinite slope of the function at  $(-10,0)$  and  $(10,0)$ . The fourth line shows the extraction of the numerator, and the fifth shows the zeros of the resulting quadratic function. Finally the sixth line shows the evaluation of the function  $a(x)$  at this list of zeros, returning a list of the corresponding values. **f\_76**

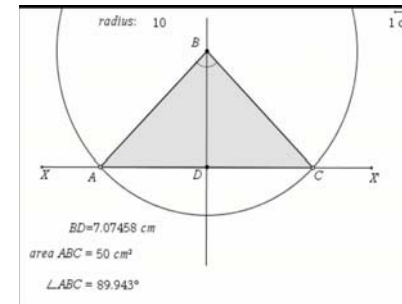


**f\_76**

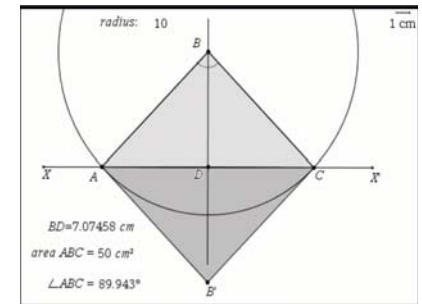
## Geometric

As this was, after all, a problem posed geometrically, it should be one which we can solve geometrically – and maybe that will also give us a greater insight into the shape of the optimum sheep pen – something which the numbers, lists, graphs, tables and symbols have so far failed to reveal! The chosen diagram starts with a section of the “wall”  $XX'$ ; a segment joining two points. The fixed point  $D$  is the midpoint of  $XX'$ . The line through  $D$  perpendicular to  $XX'$  is drawn and the slideable point  $B$  created on it. So in terms our problem we now have the point  $B$  as our **independent variable**, and its **domain** is the perpendicular bisector of  $XX'$ . The number 10 is entered as text on the screen, and used with the Compass tool and point  $B$  to construct the circle centre  $B$  radius 10 cm. (Of course the actual length in cm will depend on the size of display used!) This circle may or may not intersect the segment  $XX'$ , but where it does so we label the points  $A, C$ . Finally we construct the triangle  $ABC$  and change its attributes to fill it in. We can then measure the distance  $AD$  (i.e. the height  $x$ ) and the area  $ABC$ . As we slide  $B$  up and down so these measurements will change and we can find a good approximation to the optimum shape. **f\_77**

Because we have a geometric view now, we can see – and check – that  $AD, DC$  and  $DB$  are all more or less equal and so estimate the angles in the figure. If we now reflect  $ABC$  in  $XX'$  we get the triangle  $AB'C$ , which together with  $ABC$  makes a rhombus  $ABCB'$  of side 10cm. So the optimum configuration will occur when  $ABC$  is half a square of side 10 i.e. with an area of  $50 \text{ cm}^2$ . **f\_78**



**f\_77**



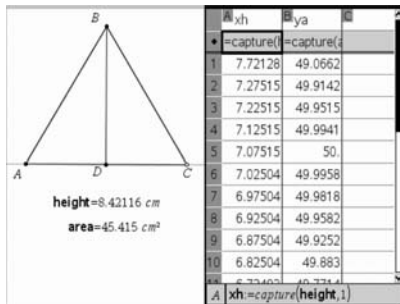
**f\_78**

There are other interesting ways in which we can represent data collected from measurements and calculations on a geometry page graphically. In the first example we will use a geometric approach to making our own graph dynamically. First we hide some of the “clutter” we have acquired such as the circle and  $AB'C$ . Using the text value 1 and the point  $D$  as origin we can use Compass to create a unit circle around  $D$ , and its intersections  $E, F$  with our new “axes”  $DX$  and  $DB$ . Creating vectors  $DE$  and  $DF$  gives us use of an important geometric tool called Measurement Transfer. The length  $BD$  lies between 0 and 10 and so we can happily Transfer it to vector  $DE$  to produce the point  $G$  on the  $x$ -axis. However the area ranges between 0 and 50 which means we need to scale it before plotting on the  $y$ -axis. So we can enter the scaling formula e.g. “ $y/10$ ” as text, and then Calculate the formula using the value of the area to substitute for  $y$ . Then the resulting, scaled, value can be transferred to the vector  $DF$  to create the point  $H$ . As  $B$  slides so the points  $G$  and  $H$  move along their respective axes. We just have to create perpendiculars to the axes through  $G$  and  $H$  to meet at the point  $J$  which is the graphical representation of the point whose coordinates are given by the height and area of the triangle  $ABC$ . So as  $B$  slides,  $J$  now traces out a path which is the graph of the area as a function of height. The set of all positions of a point such as  $J$  is called a **locus**, so we just need to construct the locus of  $J$  as a function of  $B$  to get the curve shown. The attributes of  $GJ$ ,  $HJ$  and the locus have been changed to help with clarity. Since our construction allows  $B$  to pass below  $XX'$  we have acquired a spurious branch of the locus. You might like to think how our construction could have been started differently to ensure that  $B$  is always “above”  $XX'$ .

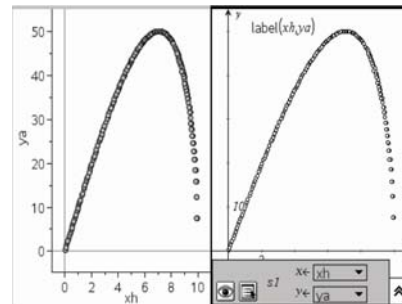
## Statistic

An alternative way to make a graph from the geometric measurements is to capture them into lists in a spreadsheet. First we copy the Geometry construction into a new Graphs & Geometry page, which we divide vertically to include a Lists & Spreadsheets area as well. We hide unnecessary parts of the construction leaving just the “bare bones” of a resizable triangle controlled by the point  $B$ , and the two measurements: length  $BD$  and area  $ABC$ . We now want to store the two measurements into variables. So using a right click with the length selected we can select the Store option and type in a name, like “height” to use for this measurement. Similarly we can store the other measurement in the variable “area”. Now we can set up the Lists & Spreadsheet to collect data automatically from these lists when we start to drag the point  $B$  about. The lists in columns A and B have been given the names  $xh$  and  $ya$ . The formula entered for  $xh$  uses the Data Capture option and then the Automated Data selection. You edit the formula  $xh:=\text{capture}(\text{Var},1)$  to replace Var with height, and similarly set up  $ya$  as the list formed by capturing area. Now as you drag the point  $B$  in the Graphs & Geometry window so the data lists grow in the Lists & Spreadsheet window. [f\\_79](#)

All we now need is to plot the data as a scattergram, which we can do either in a Data & Statistics window or a Graphs & Geometry window – or, as in the screen below, both. [f\\_80](#)

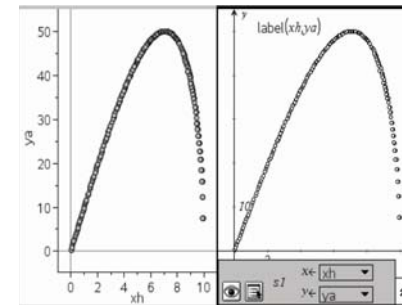


f\_79



f\_80

As a final exercise in this section we shall show how to fit a particular model to the scattergraph data. A new window is split between Lists & Spreadsheet and Graphs & Geometry. If we select Stat Calculations in the left hand window we can specify the two lists to use ( $xh$  and  $ya$ ), and the name of the function ( $f_2$ ) into which the equation will be stored. Here we see that a cubic fit has quite a close correlation, but doesn't fit close to (0,0) or (10,0) – which isn't surprising! [f\\_81](#)



f\_81

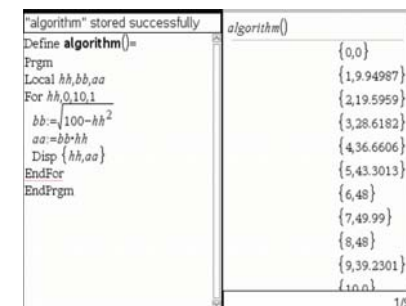
## Algorithmic

As we shall see, TI-Nspire includes a powerful programming language.

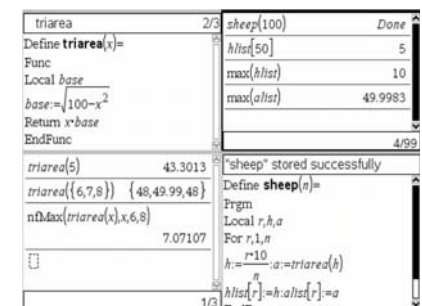
Programs and functions can be defined by inserting a Program Editor area and selecting to build a New Program. Here our program is given the name algorithm and does not pass any arguments, so it will be called e.g. from a Calculator page by entering “algorithm()”. The Editor opens up with 3 lines already completed: [f\\_82](#)

```
Define algorithm()=
Prgm
EndPrgm
```

We enter our program in the gap provided using assignment statements like  $a:=3$ , I/O statements like Display  $b$ , and structures such as For, While, If etc. Variables used in a program can either be declared to be Local, or otherwise the program will set or reset variable values which can be accessed after the program has run. Once you have entered the program you can select Check Syntax & Store, and hopefully you will get a “success” message. Once the program has been successfully entered you can use its name from a TI-Nspire page e.g. Calculator, or use it within another program definition. To illustrate the point we'll define a new function which computes the area of a triangle of given height, and then use it in a program to compute lists of heights and areas. [f\\_83](#)

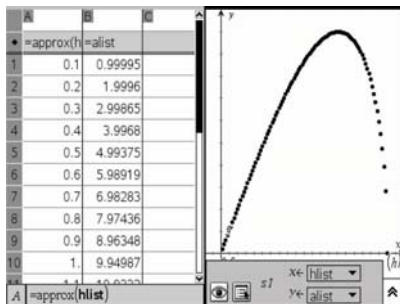
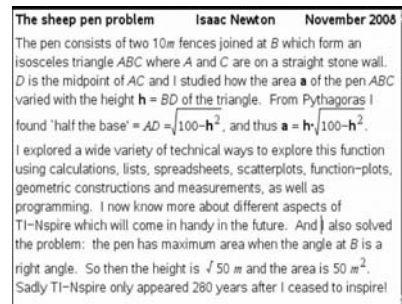


f\_82



f\_83

The function  $\text{triarrea}(x)$  has been defined using a few more lines than strictly necessary to make the point that this structure allows us to define quite complicated functions for example returning different expressions for different ranges of input variable  $x$ . When we call the function from a Calculator area, we just put in a value (constant or list) for the variable  $x$ . Now that the function has been defined we can make use of it in the program called  $\text{sheep}()$ . In order to use this you specify the value of  $n$ , which will be the number of data points generated. The program uses both local variables ( $r, h, a$ ) and also global variables ( $hlist, rlist$ ). The square brackets notation:  $hlist[r]$  means the  $r$ -th element of the list  $hlist$ . So when the program has been successfully stored we can run it with a value for  $n$  such as 100 – and then we generate the lists  $hlist$  and  $alist$  “as a by-product” of the program. These lists can then be used with other areas of TI-Nspire such as Lists & Spreadsheet or Graphs & Geometry.  $f_{84}$

 $f_{84}$  $f_{85}$ 

## Synoptic

One of the key features of TI-Nspire is that it can be used to build up a document consisting of pages each of which can be divided into 1, 2, 3 or 4 different areas. The pages can be organised within sections called “New Problem” – so that the final saved “.tns” file is really a book made up from components which are mixtures of:

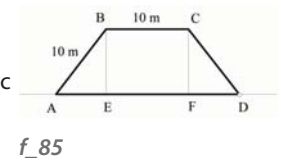
- Calculator
- Graphs & Geometry
- Lists & Spreadsheet
- Notes
- Data & Statistics
- Program Editor

Including Notes areas and/or pages allows you to express in plain text about your problem, strategy, suppositions, conclusions, next steps etc.  $f_{85}$

## Some more optimisations to try:

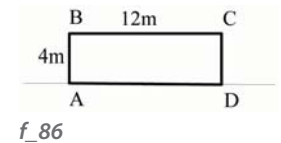
Three fences and a wall make a trapezium.

The shepherd is given an extra fence for her birthday. She knows about symmetry and she now seeks to make a symmetric trapezium ABCD of maximum area. What is this area, and how should she arrange the fences?  $f_{85}$

 $f_{85}$ 

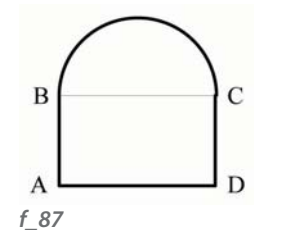
Two poles, a length of wire and a wall make a rectangle.

A neighbouring shepherd has 20 m. of wire and two poles. He decides to use the wire and the wall to make a rectangular pen. The diagram shows one possible shape where the wire is attached to the wall at A and D, and passes round the posts B and C. Find the size and shape of the biggest pen. (Can you do this using calculus as well?) Would a trapezium be better?  $f_{86}$

 $f_{86}$ 

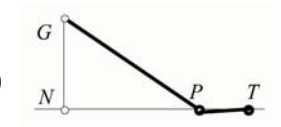
A rectangle and a semicircle make a window.

A church window is made from a rectangle with a semicircle on top. If the total perimeter is to be 10 m. find the size and shape of the window with the greatest area. (Try this with calculus as well.)  $f_{87}$

 $f_{87}$ 

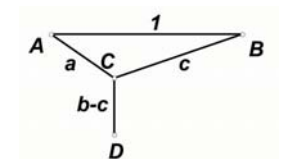
Swimming and running as fast as we can.

A girl G is swimming in a lake 100m from the nearest point N on the shore. Her towel is at a point T 200m along the (straight) shore from N. She can swim at 2 m/s and run at 5 m/s. She swims straight to a point P on the shore and then runs straight to her towel at T. What is the shortest possible time for the journey  $GP + PT$ ?  $f_{88}$

 $f_{88}$ 

Going Dutch with hanging weights.

In the Netherlands all sixth-formers have to study maths. Last year those taking the Arts & Humanities syllabus were given a project based on an historic example. The Marquis De L'Hospital (1661-1704) published the first textbook on calculus, his *Analyse des infiniment petits*, in 1696. He solved the following problem - can you? AB is a beam 1m long. AC is a rod free to pivot at A and with a freely spinning pulley at C. A rope is fixed at B, passes over the pulley at C, and hangs vertically down to a weight at D. If the length of the rod AC is 40 cm, and the length of the rope BC+CD is 1m. find the greatest depth that D can reach below AB.  $f_{89}$

 $f_{89}$

# Section 3

## Modelling – in search of realism

Another extremely important use of ICT in mathematics is for **modelling** – which might involve simulating the way a system behaves, or collecting data from it and trying to find some mathematical functions which exhibit similar behaviour. In this section we will model some simple situations in which the common theme is dynamics – i.e. a system which changes over time.

Among the key techniques we will explore are animation, parametric equations, regression, data-logging and some ideas from computer graphics. First we will start with simulating Galileo Galilei's experiments on dropping balls from the tower of Pisa in Italy – where he became Professor of Mathematics in 1589 – at the age of 25. Of course we shall have to be content to work in 2-dimensions and so our main work space will be on Graphs & Geometry pages. Here we will run into a resolvable conflict – easier to deal with than Galileo's famous conflict with the church when he went on trial in 1633, at the age of 76, and reputedly uttered the phrase: "*e pur si muove*".

In mathematics we assume that we normally do geometry on a blank page whose dimensions do not change while we are making constructions. Also, in coordinate geometry, we normally assume that the scales on each axis are equal so that the graph of  $y = x$  makes an angle of  $45^\circ$  with each axis. However in statistics, or data handling, we are used to having different scales on the axes e.g. temperature in  $^\circ\text{C}$  on the vertical axis against time in seconds on the horizontal axis to represent how the outside temperature varies with time of day, say. TI-Nspire allows us consider a Graphs & Geometry page as having two layers – one of which is a Plane Geometry View – for the creation of geometrically "correct" objects, and the other an Analytic View for the graphing of functions within which the graphs of equations change as the axis scales change.

## Animating a falling ball

Our first aim, then, is to be able to simulate dropping a ball vertically under rest – so we need to be able to animate the downward motion of a circle. In order to represent its movement analytically we will also need to draw its displacement-time and velocity-time graphs. You can make your own video of a ball dropping, or you can download one from the Internet, such as the 'avi' file from:

<http://www3.science.tamu.edu/cmse/videoanalysis/balldrop.avi>.

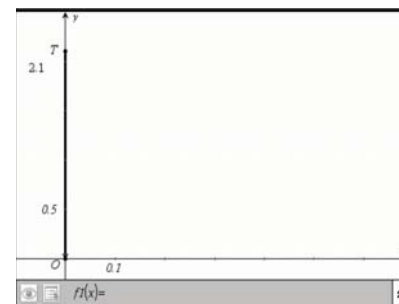
The figure below shows a video analysis of this clip showing the different positions of the ball down the y-axis and the corresponding displacement-time graph, which we will try to emulate. *f\_90*



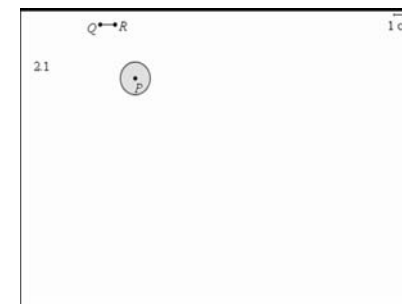
*f\_90*

So let's start with a Graphs & Geometry page, and set up axes to look like those in the video clip and analysis. Using Window Settings we have set XMin to -0.1, XMax to 0.7, YMin to -0.5 and YMax to 2.5. Using Text we have entered 2.1 and used Measurement Transfer to copy this to the y-axis where we have labelled the point as  $T$ , and then created a vector from  $T$  to the origin  $O$ . *f\_91*

The next task is to create a representation of the ball as a filled-in circle. To do this we select Plane Geometry View, and create a point  $P$  and segment  $QR$ . Using the Compass tool we can construct the circle centre  $P$ , radius  $QR$  and adjust its Attributes to be filled-in with grey. *f\_92*



*f\_91*



*f\_92*



Now we are ready to merge the two views. So we select Graphing View, and reset the Window Settings to our chosen values. We can now create a point  $B$  on the vector  $TO$  and use Compass to construct a geometrically correct circle centre  $B$  radius  $QR$ . However the snag is that we want  $B$  to be the bottom of the ball, so we find the intersection  $C$  of the circle with the  $y$ -axis above  $B$ , define the vector  $BC$  and translate the circle by  $BC$ . Now we can shade it in and we have a “ball” which we can slide up and down by dragging point  $B$ . (Of course there are other ways of creating this circle – e.g. using Compass again with  $C$  as centre and  $PQ$  as radius, or Redefining the point  $P$  to become  $C$ .) *f\_93*

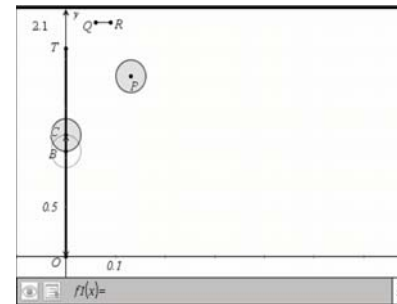
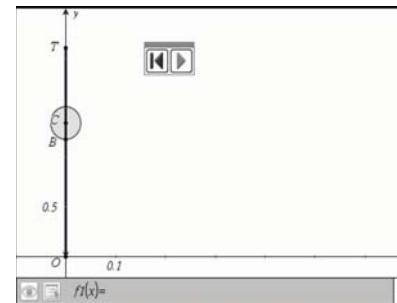
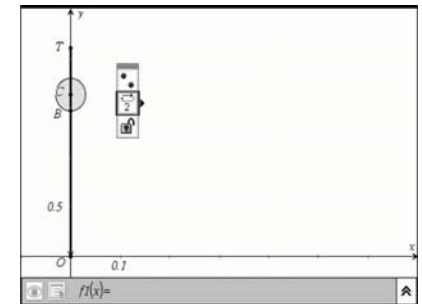
We can animate the point  $B$  on  $TO$  by changing the second of its Attributes – which controls Uni-directional animation speed. When the speed e.g. 2 has been entered then the “ball” goes into motion repeatedly falling down vector  $TO$ . The motion controller allows us to pause and restart it. *f\_94-95*

So while we have now have something dynamic, it doesn’t actual simulate the way the ball drops, since it produces motion with constant speed – while Galileo found that the ball’s speed was continually changing. So this is where time comes in!

In order to understand this better we can build a spreadsheet to model the fall under constant acceleration  $g \approx 9.8 \text{ ms}^{-2}$ . Here we use column B to define some constants:  $g$  (acceleration due to gravity),  $dt$  (time step) and  $bh$  (ball’s initial height) – and column A to show their names. Column C holds  $bt$  (ball time) defined as the sequence of 8 terms starting at 0 in steps of  $dt$ . Column D holds  $bv$  (ball velocity) defined as  $g*bt$ . Column E holds  $bd$  (ball drop distance) calculated by adding the average velocity in the next step multiplied by  $dt$  to the current drop distance, starting from zero. So this has the recurrence format:  $e2 = e1 + (d1+d2)/2*dt$  which is copied down column E. Column F holds the ball’s height by calculated by subtracting  $bd$  from  $bh$ . (Notice that TI-Nspire checks for each appearance of a variable whether you want to you use a defined variable – like  $g$  – or the column with that name.) Finally column G holds  $pred$ , the predicted height, using the formula  $bh - 1/2*g*dt^2$ . *f\_96*

So after the numerical diversion we can get back to putting the two key ideas together – that of animating a point on an object which moves at a constant rate, and that of using the Galilean equation of motion to determine the position of the ball at a given time  $t$ . To do this we will make use of some of the constants we defined on the Spreadsheet:  $g$  and  $bh$ . So we can copy and paste our earlier Graphs & Geometry page and this time delete the point  $B$  on vector  $TO$  which we were using as our independent variable.

We can also use Hide/Show to bring back any “tidied-up” detail we may need to use again, such as segment  $QR$ . The first job is to calculate the time at which the ball will hit the ground – and this is when  $bh - 1/2*g*t^2 = 0$ . Using the Text tool we can enter the formula  $\sqrt{(2*bh/g)}$  and then use the Calculate tool to evaluate this – pressing ‘shift’ and ‘L’ to recapture the stored values for each of  $bh$  and  $g$ . The computed value of can rounded

*f\_93**f\_95**f\_94*

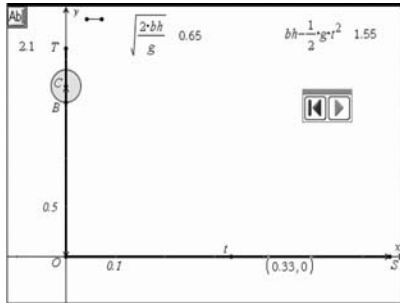
		bt	bv	bd	by	pred	
1	g	9.8	0.	0.	0	2.1	
2	dt	0.1	0.98	0.049	2.051	2.051	
3	bh	2.1	0.2	1.96	0.196	1.904	1.904
4		0.3	2.94	0.441	1.659	1.659	
5		0.4	3.92	0.784	1.316	1.316	
6		0.5	4.9	1.225	0.875	0.875	
7		0.6	5.88	1.764	0.336	0.336	
8		0.7	6.86	2.401	-0.301	-0.301	
9							
10							
E2	$=e1 + \frac{d1+d2}{2}*dt$						

*f\_96*

(using Attributes) to 2 d.p. to give 0.65. Using Measurement Transfer this value can define the point labelled  $S$  on the  $x$ -axis.

The vector  $OS$  is created to represent the duration of the fall and a point  $t$  created on it to represent the time of fall  $t$  seconds. With Coordinates & Equations we can measure the coordinates of the point  $t$ . We can now work out the corresponding height of the ball using the formula  $bh - 1/2*g*t^2$ . Again the values of  $bh$  and  $g$  will be stored ones from the spreadsheet (so use shift-L) whereas  $t$  will be the  $x$ -coordinate of point  $t$ . With Measurement Transfer use this to define the point  $B$  on the  $y$ -axis (not the vector  $TO$ ). Again we can use Compass to construct the circle radius  $PQ$  centre  $B$ , and find its intersection  $C$  with the  $y$ -axis (not the vector  $TO$ ) to define the ball.

Now we can check that point  $t$  on  $OS$  is the new independent variable, and that the point  $B$  moves as the dependent variable on the  $y$ -axis. All that remains is to change the Attributes of point  $t$  to put it into Unidirectional animation with a given speed. So now we can reset the clock by dragging  $t$  to  $O$ , and start the clock by pressing the “go” button on the Animation Controller. Then the ball will “fall” vertically in simulated time showing how it accelerates due to the gravitational attraction of the Earth. *f\_97*



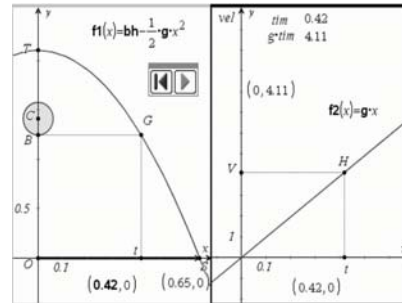
f\_97

As the final step in the falling-ball simulation we can divide our Graphs & Geometry into a second page on which we can illustrate the change in velocity of the ball. On the left screen we can right click on the  $x$ -coordinate of point  $t$ , say, and Store it into a variable called **tim**, say. We can also use Hide/Show to tidy that screen up. f\_98

In order to illustrate the concept of the displacement-time graph we can define the function  $f1(x)$  and plot its graph. We can also construct perpendiculars to the axes at  $t$  and  $B$  and also their intersection point  $G$ , showing that  $G$  tracks the graph when  $t$  is put in motion. On the right-hand screen we can adjust axes and plot the graph of the velocity-time function in  $f2(x)$ . We can use the variable **tim** to control the constructions here by entering "tim" as text, and Calculating its value using shift-L to recapture the value from the left-hand page. This number is Measurement Transferred to the  $x$ -axis to give point  $t$ , a formula is entered and calculated to find the value of the corresponding velocity and that is Measurement Transferred to the  $y$ -axis to give point  $V$ . Then Perpendiculars through  $t$  and  $V$  are constructed, together with their intersection  $H$ .

Now, as the animation is controlled from the left-hand screen you should see both sides of the screen showing animations – with distance fallen on the left and velocity achieved on the right.

Sadly Galileo missed out on getting his hands on TI-Nspire by some 365 years!



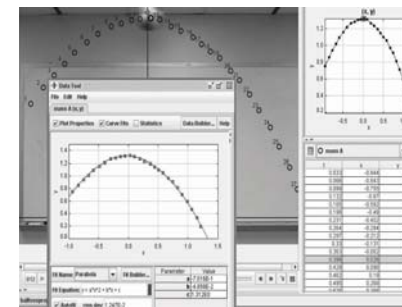
f\_98

## Simulating a projectile

You might like to try extending the techniques of the last section to modelling the motion of a ball being thrown. Here we can use three variables, or co-ordinates, to represent the position of the ball  $(x, y, t)$ . The graph of  $y$  against  $x$  is the path of the ball in spaces, known as its **trajectory**. We can also study the velocity-time and displacement-time graphs for the ball in either the vertical direction (like Galileo's dropped ball) or in the horizontal direction (like Isaac Newton going skating on the icy fens). You can make your own video of a ball being thrown, or find one on-line such as at:

<http://www3.science.tamu.edu/cmse/videoanalysis/balltossproj.avi> . f\_99

The key thing about a projectile is that once it has been launched it is not powered in any way. Whether or not it will be well modelled by the Galileo/Newton approach depends upon a number of factors. For example a shuttlecock at badminton behaves very differently from a tennis ball, a golf ball, a cricket ball, a hammer-throw or a shot-put. So at times we have to consider other factors such as spin, wind, air-resistance, rotation etc. But we'll start with the simplest model first! So let's assume that the only force being applied is a due to the constant vertical acceleration downwards through the Earth's gravitational attraction. We'll also assume that the projectile is launched from, and returns to, ground level – and that the ground is indeed flat! You might like to record some data for yourself. A simple rocket kit, like the Super STOMP one illustrated below, gives a fun way of gathering data. f\_100



f\_99



f\_100

A good jump or stamp on the box will give it enough thrust to cover a range of c100 m. In fact, if we can ignore factors such as wind, spin, air-resistance etc. the path is completely determined once we know the launch velocity  $L \text{ ms}^{-1}$  and launch angle  $\alpha^\circ$  together with the acceleration due to gravity  $g \text{ ms}^{-2}$ . Using the Galileo/Newton approach we can calculate pretty well all we need to know about the trajectory from estimates or measurements of the horizontal range  $R \text{ m.}$  and the flight duration  $F \text{ s.}$  together with the angle  $\alpha^\circ$ . Horizontally the velocity will be constant at  $H \text{ ms}^{-1}$ , say, where  $H = R/F$ . **f\_101**

The launch velocity has horizontal component  $H$  and vertical component  $V$ , say.

So  $H = L \cos \alpha$  and  $V = L \sin \alpha$ , giving  $L = H/\cos \alpha = R/(F \cos \alpha)$  and  $V = H \tan \alpha = (R/F) \tan \alpha$ . Vertically we know that the rocket's final speed will be the same as its initial speed but in the opposite direction. So in time  $F$  its velocity will have changed from  $+V$  to  $-V$ , with an acceleration given by  $g = (V - (-V))/F$ , so  $g = 2V/F = (2R/F^2) \tan \alpha$ . In time  $F/2 \text{ s.}$  it will have reached its maximum height  $M \text{ m.}$ , when its vertical velocity will have become 0. Its average speed is  $V/2 \text{ ms}^{-1}$  and thus  $M = (V/2)(F/2) = (R/4) \tan \alpha$ . So we can even calculate an estimate of  $g$  from our 3 basic measurements of a length  $R$ , a time  $F$  and an angle  $\alpha$ . We are now in a position to set up a spreadsheet in which we can enter our three measurements and calculate the other useful data. Column C is not needed, but just entered, like column A, with quotation marks to show the formulae being used. These are embedded in column B. An additional constant  $k$  has been calculated – so that  $y = kx(R-x)$  is the Cartesian equation of the trajectory. **f\_102**

*Note:* we use upper and lower cases for variables interchangeably – so that  $f$  and  $F$  both refer to the same variable: the flight duration.

With the data entered we can divide the screen and introduce a Graphs & Geometry region in which we can plot the parametric graph of the trajectory. The value of  $t$  runs from 0 to  $f$  in steps of  $f/20$ . You may need to drag the scales to fit the flight path to the available area – and then use Zoom Square to check that the picture is geometrically accurate. Values of  $m$  and  $r$  have been recalled to the graph screen and used with Measurement Transfer to create the points  $R$  and  $M$ . Similarly values of  $h$  and  $v$  have been recalled and used to create the vectors  $OH$  and  $OV$ , as well as their resultant  $OL$ . Now we have all we require to simulate the actual flight.

First we can resize the windows so that we have a larger graphic area, and then use Zoom Square again to ensure realism. The point  $x$  is created on the segment  $OR$  to control the animation. Its coordinates are measured and used to calculate the corresponding value of time  $t$ . This is used to calculate the  $y$ -coordinate of the point  $S(x,y)$  which represents the position of the STOMP rocket. The rocket's horizontal velocity at  $S$  remains constant, and we can compute its current vertical component which we represent by the point  $vy$  on the  $y$ -axis. Translating by vector  $OS$  allows us to construct the velocity vector at  $S$  – which is tangent to the flight path.

Then we can select the Attributes of point  $x$  and choose its Unidirectional animation speed and we have completed the task of simulating the idealised flight of the rocket from just our three basic measurements: range, time and angle.

Now the advertising on the STOMP box claims that the rocket will travel “up to 400 feet”. The distance actually travelled by the rocket is the path length along the curve from  $O$  through  $M$  to  $R$ . In mathematics this is called the **arc-length** and to calculate it requires knowledge of the calculus techniques of integration. There are other ways we could approximate this calculation. For example if we printed out the graph, we could cut a piece of string to fit the curve and then convert it to a measurement in metres knowing the scale of the print-out. We could also take a number of successive points on the graph and sum the distances between them – to give a polygonal approximation which be more accurate as we increase the number of points used.

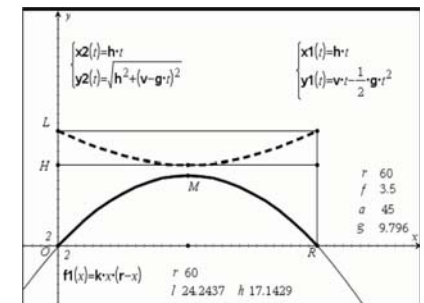
An alternative approach is to find the average speed  $S \text{ ms}^{-1}$  over the flight, so that the distance  $D \text{ m.}$  flown will then be given by  $D = S.F \text{ m.}$  The maximum speed is  $L$  at  $t=0$  and  $t=F$ , and the minimum speed is  $H$  at  $t=F/2$ . So a working approximation to the average speed is given by  $S \approx (L+H)/2$  so that  $S \approx R/(2F) (1 + \sec \alpha)$  and  $D \approx R(1 + \sec \alpha)/2$ . So, in the case above we have the approximate distance flown as  $D \approx 68.3 \text{ m} \approx 230 \text{ ft.}$  To get a better approximation we can graph the function which shows the speed at any time and position. To make the graphs clearer we've reduced the values of  $r$  and  $f$  to 60 m. and 3.5 s. respectively.

*Note:* when you change  $f$  you have manually to update the range of the  $t$  variable in any parametric equations.

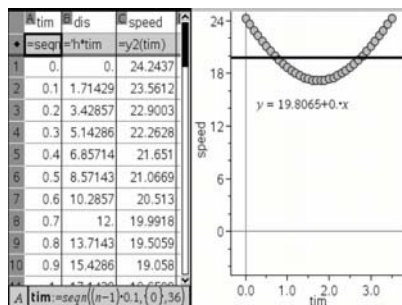
In a new Graphs & Geometry page you can plot both the Cartesian form of the trajectory and the equivalent parametric form. The advantage of the parametric one being that it just plots for the range of  $t$  in which we are interested i.e.  $[0, f]$ . The dashed graph is the parametric curve representing the average speed  $S$  as a function of time  $t$ . As anticipated it varies in height between  $H \leq S \leq L$ . **f\_102**



**f\_101**



**f\_102**



f\_103

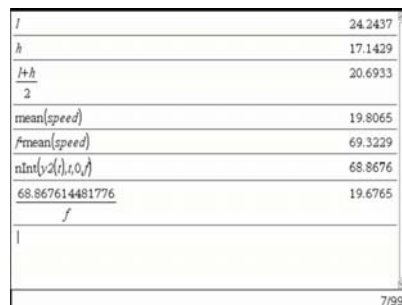
We can open a new Lists & Spreadsheets page to create three lists. The list "tim" uses the Generate Sequence command  $\text{seqn}((n-1)*0.1, \{0, 36\})$  to put a column of times  $\{0, 0.1, 0.2, \dots, 3.5\}$  in *tim*. The list "dis" is just the product of the horizontal velocity  $h$  and the time. The list "speed" uses the parametric function  $y2(\text{tim})$  to return the current speed at each interval of *tim*. Dividing the screen and displaying a Data & Statistics window we can plot the scattergram of *speed* against *tim*. f\_103

The resulting linear regression function shows that the approximate average speed  $S'$  during the flight is  $S' = 19.81 \text{ ms}^{-1}$ , compared with our rough estimate of  $(L + H)/2 = 20.7 \text{ ms}^{-1}$ . So the distance flown is  $D = SF = 69.32 \text{ m}$ .  $\approx 210$  feet. Our value for  $S$  is approximate because we are taking a finite number of discrete data values. In a Calculator screen we can use the Numerical Integral function  $\text{nInt}(y2(t), t, 0, f)$  to find the distance flown  $D$  as the area under the distance time graph  $y = f2(t)$  between  $t=0$  and  $t=f$  i.e.  $D = 68.87 \text{ m}$ . Dividing this by the flight time  $f$  we find that the average speed was  $S \approx 19.68 \text{ ms}^{-1}$ . f\_104

Can you find the minimum launch velocity  $L$ , and corresponding launch angle  $\alpha$  for which  $D = 400 \text{ feet} \approx 122 \text{ m}$ ?

## How do the planets move, Mr. Kepler?

The Italian mathematician, Galileo Galilei (1564-1642), and the English mathematician, Sir Isaac Newton (1643-1727) made significant contributions to the way we apply mathematics to physical situations. Our modern theories of planetary motion also go back a long way in history – probably starting with the Polish astronomer, Nicolaus Copernicus (1473-1543), who proposed a model of the solar system in which the planets orbit the Sun, rather than having the Earth at centre of the system. The Danish astronomer Tycho Brahe (1546-1601) made meticulous observations of planetary positions which were drawn on by the Austrian mathematician Johannes Kepler (1571-1630) in postulating his three "Laws" of planetary motion. You can find more about these at the following url: <http://csep10.phys.utk.edu/astr161/lect/history/kepler.html>.



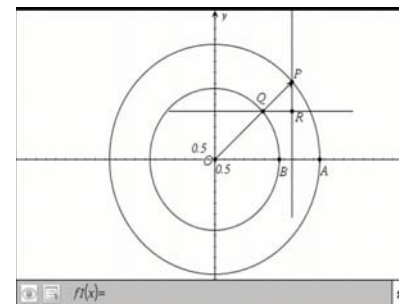
f\_104

The first law states that planets orbit the Sun in ellipses with the Sun as one focus. So we need to review what we can find about ellipses. One method of constructing an ellipse uses a pair of concentric circles, another uses parallel chords of one circle, a third uses two fixed points and a length ("string and 2 pins") and a fourth uses a fixed point and a fixed line. From these we can also derive various forms of the coordinates of points on the ellipse: Cartesian, parametric and polar.

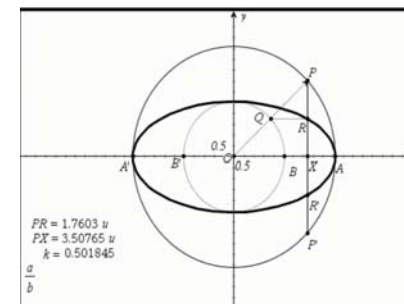
## Ellipses through geometry

On a new Graphs & Geometry page first define a point  $A$  on the  $x$ -axis and the point  $O$  as the intersection of the axes. Then create the segment  $OA$  and a point  $B$  on it. Construct the circle centre  $O$  through  $A$  and through  $B$ . Construct the point  $P$  on the circle through  $A$  and construct the vector (or segment)  $OP$ . Create  $Q$  as the point of intersection of  $OP$  with the circle through  $B$ . Create perpendicular to the axes through  $Q$  and  $P$  to meet in  $R$ . The path of  $R$  as you drag (or animate)  $P$  on the circle is the required ellipse. f\_105

You can construct the ellipse as the locus of  $R$  with  $P$ , and then tidy up the figure. You can see the effect on the shape of the ellipse when you slide  $B$  on  $OA$ . If  $X$  is the foot of the perpendicular from  $P$  to the  $x$ -axis we can measure the distances  $PR$  and  $PX$ , and see that their ratio  $PX/PR = k$  stays constant as you drag  $P$ . The circles through  $A$  and  $B$  are called the **major** and **minor auxiliary circles**. f\_106



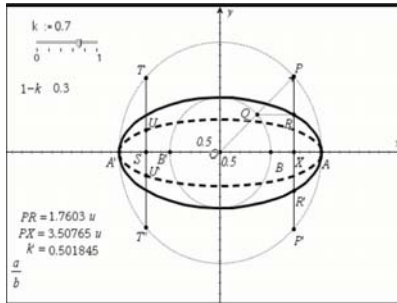
f\_105



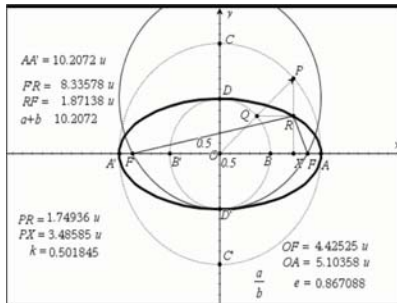
f\_106

This "discovery" provides our second method of drawing an ellipse. Take any point  $S$  on the diameter  $AA'$  of the circle centre  $O$  and construct the chord  $TT'$ . Define a variable  $k$  between 0 and 1 by the use of a slider. Use Text and Calculate to make the value of  $k$  available on the Graph display. Then dilate the chord  $TT'$  with centre  $S$  and scale factor  $1-k$  to produce the chord  $UU'$ . The loci of  $U$  and  $U'$  with  $S$  together form the ellipse. Now you can slide  $k$  to see that when it agrees with the ratio, now called  $k'$ , of the first ellipse, then the two loci are identical – as we would expect. f\_107

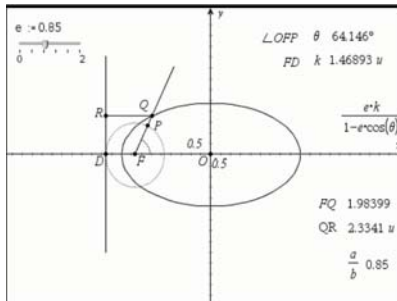




f\_107

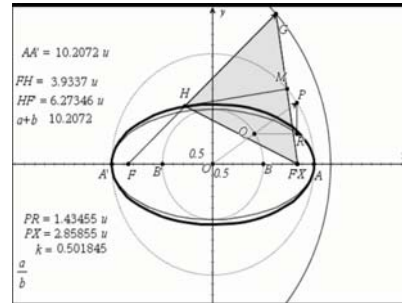


f\_109

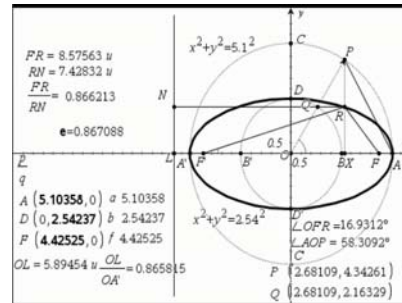


f\_111

In order to see the construction approaches together we need to be able to find the locus of a point  $H$  which moves so that the sum of its distances from two fixed points  $F, F'$  is a constant  $d > FF'$ . We can construct the point  $F$  on the segment  $AA'$  and choose the length  $AA'$  (the diameter of the major circle) as the constant  $d$ . In order to keep symmetry about  $O$  we just reflect  $F$  in the  $y$ -axis to construct  $F'$ . Now we can use the Compass tool to construct a circle centre  $F$  with radius  $= AA'$ . If we take  $G$  as a point on this circle we can construct a radius  $FG$ . We want to locate a point  $H$  on  $FG$  for which  $HF' = HG$ , since then  $FH + HF' = FH + HG = FG = AA'$ . This means that we need to construct an isosceles triangle with  $F'G$  as base. So  $H$  is the intersection of the perpendicular bisector of  $F'G$  with  $FG$ .



f\_108



f\_110

The locus of  $H$  with  $G$  is then the required ellipse, and we can change its shape by moving  $F$  nearer to, or further from,  $O$ . The points  $F, F'$  are called the **foci** of the ellipse. f\_108

In order to identify the exact positions of  $F, F'$  on  $AA'$  we can look at the symmetric case when  $P$  is at  $D$ , the intersection of the minor circle with the  $y$ -axis. Then  $FDF'$  is an isosceles triangle with  $F'D = DF = OA$ . So  $F, F'$  are the intersections of the circle centre  $D$  radius  $OA$  with  $AA'$ . We can then measure  $PF$  and  $PF'$  and check that  $F'P + PF = AA'$  as  $P$  describes the major circle. The segments  $AA'$  and  $DD'$  are called the **major** and **minor axes** of the ellipse. The lengths of these are called  $a$  and  $b$  respectively. The ratio of the distances  $OF$  and  $OA$  is called  $e$ . The value of  $e$  varies when you drag  $B$ , and it is called the **eccentricity** of the ellipse, which is a number between 0 and 1. The actual eccentricity of most planets, especially the Earth, is not much greater than 0 – so that its orbit is not far from being circular. f\_109

So far we haven't made any use of co-ordinates, but we now we can start to gather some data. The new point  $L$  has been created on the  $x$ -axis, and the perpendicular drawn through it. The foot of the perpendicular from  $R$  to this line is  $N$ . The ratio  $F'R/RN$  has been calculated. We can drag  $L$  to try to make this match  $e$  as closely as possible for the current position of  $P$ . Then as  $P$  is dragged round the major circle we can observe that the ratio is more or less constant. In fact when the ratio of  $OL$  to  $OA'$  is exactly  $e$  then  $F'R = e \cdot RN$ . The line  $LN$  is known as a **directrix** of the ellipse, and locus of any point  $R$  satisfying the rule that the ratio of its distance from a fixed point ( $F'$ ) to its distance from a fixed line ( $LN$ ) is a constant ( $e$ ) is called a **conic** or a **conic section**. In the case where  $0 < e < 1$  the conic is called an **ellipse**, when  $e = 1$  it is a **parabola** and when  $e > 1$  it is a **hyperbola**. f\_110

This gives us another construction, based on measurements, which will generate any of the conics. On a new Graphs & Geometry page create points  $F$  and  $D$  on the  $x$ -axis. Construct the circle centre  $F$  through  $D$  and any point  $P$  on it. Measure the angle  $OPF$  and call it  $\theta$ . We need to find the point  $Q$  on  $FP$  for which  $FQ/QR = e$ , the eccentricity. We can create a slider with which to set the value of  $e$ , and we can measure the distance  $FD$  and call it  $k$ . So, if  $FQ = r$ , then  $RQ = DF + FP \cos(\theta) = k + r \cos(\theta)$  and then  $FQ = r = e \cdot QR = e(k + r \cos(\theta))$ , so  $r(1 - e \cos(\theta)) = e \cdot k$ , giving  $r = e \cdot k / (1 - e \cos(\theta))$ . So you can enter this formula for  $r$  as text, and then Calculate its value using the current values of  $e$ ,  $k$  and  $\theta$ . The final steps are to use **Measurement Transfer** with the ray  $FP$  to construct  $Q$ , and then construct the locus of  $Q$  with  $P$  to give the required conic. Now you adjust the value of  $e$ , and the positions of the focus  $F$  and directrix  $D$  at will to see how the shape of the conic is affected. f\_111

So now we are in a position to see how these varied approaches to the geometric modelling can be turned into algebra.



## Ellipses through algebra and coordinates

We'll start with the auxiliary circles. These are of the form  $x^2 + y^2 = r^2$  where  $r$  is the radius. This circle passes through  $(r, 0)$ ,  $(0, r)$ ,  $(-r, 0)$  and  $(0, -r)$ . The ellipse passes through  $(a, 0)$ ,  $(0, b)$ ,  $(-a, 0)$  and  $(0, -b)$  and so satisfies the equation  $(x/a)^2 + (y/b)^2 = 1$  – which we will soon prove to be the **Cartesian equation** of an ellipse. Our normal system of co-ordinates is known as a Cartesian system in honour of the French mathematician René Descartes (1596-1650) who is credited with making the link between algebra and geometry. In order to plot this curve as a function  $y$  of  $x$  we need to transform the equation into:  $y = \pm b \sqrt{1 - (x/a)^2}$  – which we need to draw separately first as a positive branch  $f_1(x)$  and then as a negative branch  $f_2(x)$ . [f\\_112](#)

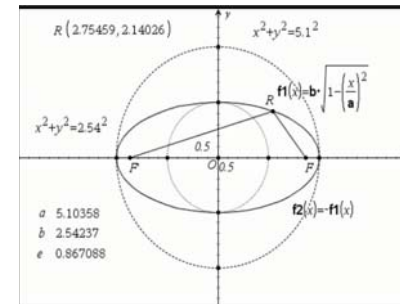
Now any point  $P$  on the major circle can be written in the form  $(a \cos(t), a \sin(t))$ , and the corresponding point  $Q$  on the minor circle is  $(b \cos(t), b \sin(t))$ . As the point  $R$  on the ellipse has the  $x$ -coordinate of  $P$  and the  $y$ -coordinate of  $Q$  then its coordinates are  $(a \cos(t), b \sin(t))$  – and these are the **parametric equations** of the ellipse. So on a Graphs & Geometry page you can select Parametric and enter the equation  $x = a \cos(t)$  and  $y = b \sin(t)$  – and also the range of values for  $t$ . If you have selected Degree mode then this will be for values of  $t$  from 0 to 360 in steps of say 5 degrees. You can check that, for example, a range of 90 to 270 for  $t$  gives you just the left half etc.! [f\\_113](#)

Since  $\sin^2 t + \cos^2 t = 1$ , we have also proved that:  $(x/a)^2 + (y/b)^2 = 1$ .

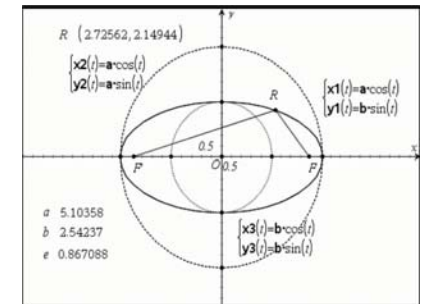
Finally we can see how the focus-directrix definition can be translated into the **Polar equation** of the ellipse. Since this will be centred on the focus as origin:  $r = r_1(\theta)$ , we will have to apply a translation to shift the graph to match the geometric construction. [f\\_114](#)

So now here is some good and bad news about ellipses. We know the area of a circle is  $\pi r^2$  and its circumference is  $2\pi r$ . The area of an ellipse is  $\pi ab$ , but there is no exact formula for its perimeter. The collective name for the circle, ellipse, parabola and hyperbola is the **conic sections** (or **conic**, for short) – which describe the various curves formed when a plane cuts a (double) cone – in 3 dimensions. Blaise Pascal (1623-62) is another famous French mathematician. He devised a construction for the conic which passes through any 5 given points – see if you can find out about Pascal's Hexagon, and how it can be used to construct conics. [f\\_115](#)

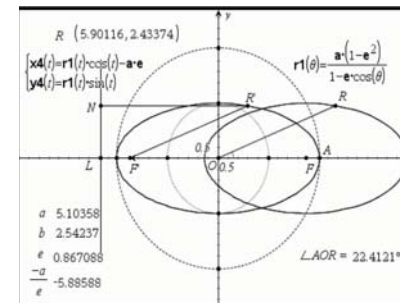
Again we leave it for you to find out more about Kepler's second law – and to see if you can use it to animate a point on an ellipse to simulate the motion of a planet around the Sun, or an electron around the nucleus of an atom.



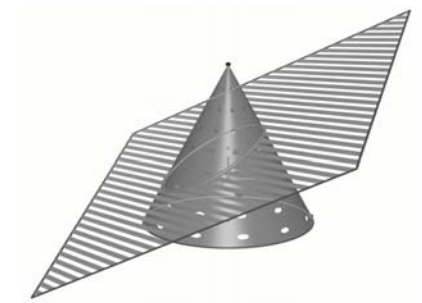
f\_112



f\_113



f\_114



f\_115

## Kepler's third law

We will see if we can use our modern technology to derive a relationship between a planet's mean distance from the Sun and the time taken to complete an orbit of the Sun (its **period**). First we need to find some data about the planets in the Solar system to give us values for the distances and periods. The table below contains these, and other data, such as the eccentricity. [f\\_116](#)

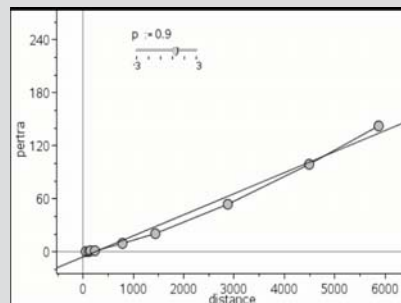
Planet	Mean Distance From The Sun ( $\times 10^6$ km)	Mean Distance From The Sun (Earth = 1)	Period To Revolve Around The Sun	Mass of planet (Earth=1)	Orbital Eccentricity
Mercury	57.91	0.387	88.0 days	0.0553	0.2056
Venus	108.21	0.723	224.7 days	0.815	0.0067
Earth	149.6	1.0	365.25 days	1.0	0.0167
Mars	227.92	1.524	687.0 days	0.107	0.0935
Jupiter	778.57	5.204	11.75 years	317.83	0.0489
Saturn	1,433.53	9.582	29.5 years	95.159	0.0565
Uranus	2,872.46	19.201	84 years	14.536	0.0457
Neptune	4,495.06	30.047	165 years	17.147	0.0113
Pluto	5,869.66	39.236	248 years	0.002	0.2444

f\_116

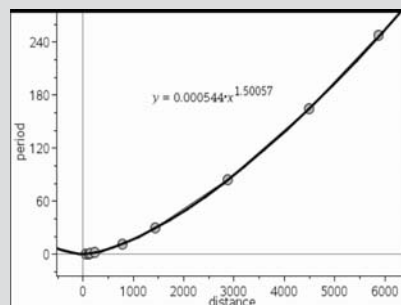
planet	distance	period
Mercury	57.91	0.240931
Venus	108.21	0.615195
Earth	149.6	1.
Mars	227.92	1.8809
Jupiter	778.57	11.75
Saturn	1433.53	29.5
Uranus	2872.46	84.
Neptune	4495.06	165.
Pluto	5869.66	248.

Planetary data from Solar system:  
Distance is mean distance from the Sun in millions of km.  
Period is the time taken to orbit the Sun in earth years = 365.25 days = 3.15576e7 secs  
Kepler's 3rd law relates the period to the distance – can we derive a model from this data?

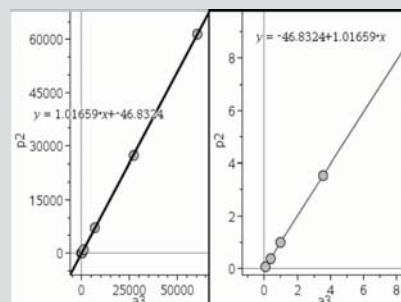
f\_117



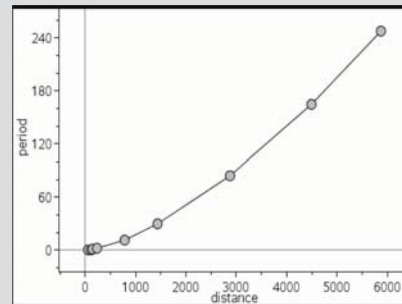
f\_119



f\_121



f\_123



f\_118

planet	distance	period	pertra
Mercury	57.91	0.240931	0.251441
Venus	108.21	0.615195	0.624227
Earth	149.6	1.	1.
Mars	227.92	1.8809	1.84559
Jupiter	778.57	11.75	10.9128
Saturn	1433.53	29.5	26.6518
Uranus	2872.46	84.	73.5446
Neptune	4495.06	165.	141.566
Pluto	5869.66	248.	210.193

Planetary data from Solar system:  
Distance is mean distance from the Sun in millions of km.  
Period is the time taken to orbit the Sun in earth years = 365.25 days = 3.15576e7 secs  
Kepler's 3rd law

f\_120

period	distance	p2	d3	k
0.240931	57.91	0.058048	194205	2.98899e-7
0.615195	108.21	0.378465	1,267,076	2.98692e-7
1.	149.6	1.	3,348,076	2.98679e-7
1.8809	227.92	3.5378	1,183,997	2.98804e-7
11.75	778.57	138.063	4,719,478	2.92538e-7
29.5	1433.53	870.25	2,945,929	2.95409e-7
84.	2872.46	7056.	2,370,071	2.97712e-7
165.	4495.06	27,225.	9,082,521	2.99752e-7
248.	5869.66	61,504.	2,022,271	3.04134e-7

f\_122

$\frac{(1.496 \cdot 10^{11})^3 \cdot 4\pi^2}{(3.16 \cdot 10^7)^2 \cdot 1.989 \cdot 10^{30}}$	6.65496e-11
Ans	1/99

Newton used calculus to prove Kepler's laws and derived the relationship:  $P^2 = k D^3$  where  $P$  is period,  $D$  is distance and  $k = \frac{4\pi^2}{G(M+m)}$  – so can we use the Earth's data to find the gravitational constant  $G$ , where  $M$  is mass of the Sun and  $m$  is the mass of the Earth?  
 $P = 3.16 \text{ E7 s}$ ,  $D = 1.496 \text{ E11 m}$  and  $M = 1.989 \text{ E30 kg}$  and  $m \ll M$ :  
so:  $G = 6.6 \text{ E-11 m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ .

f\_124

Using a Lists & Spreadsheets page we can enter some of this data. Names, such as Mars, are entered in quotation marks: "Mars". f\_117

In a Data & Statistics page we can create the scattergraph for the data, and with a right-click on the graph we can bring down the context menu and select the option to show connecting lines. While this helps us to visualise the underlying function, the actual graph is meaningless since we know there aren't any "in-between" planets! f\_118

It doesn't appear that there is a simple linear relationship between distance and period. We can explore whether distance is proportional to some power  $p$  of the period by introducing a slider to vary the value of  $p$ , together with a new data column, called "pertra" – short for "period transformed", defined as  $\text{period}^p$ . The scattergram of pertra against distance can now be used for experimentation as  $p$  is changed. It appears that when  $p$  is around 0.7 that the scattergram becomes almost linear. f\_119

If we go back to the original scattergram we can fit one of a variety of regression models – and the appropriate one here is Power Regression. The resulting power is remarkably close to  $1.5 = 3/2$ . Which suggests that if we had used  $p = 2/3 \approx 0.66$  then we would have found a very close fit to a linear function. f\_120

In a way this sounds as if there could be a stronger version of Kepler's 3<sup>rd</sup> law: that the ratio of the square of the period of any planet to the cube of its average distance is a constant. We shall see that this is nearly, but not quite, the case. If we divide the period squared by the distance cubed we get nearly a constant value  $k$ . f\_121

We can display the data on adjacent windows using different scales to see that there is a good fit for both the near and the far planets. f\_122

Newton found that the ratio  $k$  was not a constant, but a value which depends upon the mass of the planet. If we measure in SI units (seconds, metres and kilograms) then

$$k = \frac{\pi^2}{G(M+m)}$$

The Earth's period is one year  $\approx 365.25$  days  $\approx 365.25 \times 24 \times 60 \times 60 \approx 3.16 \times 10^7$  seconds. Its mean distance is  $149.6 \times 10^6$  km  $= 1.496 \times 10^{11}$  m. The Sun's mass is  $M = 1.989 \times 10^{30}$  kg, and that of the Earth is  $m = 5.9742 \times 10^{24}$  kg which is much smaller than  $M$ . f\_123

So, ignoring the Earth's mass, we can find a good approximation for Newton's "gravitational constant" as  $6.673 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \approx 6.6 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ . See what else you can find out about gravitational attraction between bodies, as well as orbits of satellites. What can you find about escape velocity? f\_124

## Modelling with live data

We have used video capture and algebra to keep track of our moving objects – balls and rockets. TI-Nspire has been designed to be used with a variety of data-capture devices from which we can take direct measurements. One key such device is the CBR2, which records the distance to an object at regular intervals by timing the emission and reflection of an ultra-sound pulse. Using the CBR2 we will record data for the intrepid bungee jumping elephant. Here we have set up the CBR2 to record data at intervals of 0.02 sec for 5 seconds directly into a Data & Statistics page. The data from the first run are stored into lists such as `dc01.time` and `dc01.dist1`. From these numerical approximations to velocity and acceleration are stored in `dc01.vel1` and `dc01.acc1`. Data for the next run would be stored as e.g. `dc02.time` etc. [f\\_126](#)

We can open a Lists & Spreadsheets page, and set up columns called  $t$ ,  $d$ ,  $v$  and  $a$  to hold the collected data. [f\\_127](#)

We can show the data in a variety of ways. For example in a Data & Statistics page we can show the graphs of distance, velocity and acceleration against time all on the same scales. Since the accelerations are computed from the velocities which are themselves computed from the displacements we can see some rather “noisy” inaccuracies appearing. However we can see that successive maxima of the displacement correspond to zeros of the velocity, and (more or less) minima of the acceleration. [f\\_128](#)

We can also superimpose a function over the data and experiment with changing the parameters to get a good fit ‘by eye’ – from which we can estimate the frequency and period, as well as mean value. [f\\_129](#)

In fact there is a built-in ‘sinusoidal’ regression function. However this works in radians, whereas we have been using the mode setting for degrees. If we convert using  $1 \text{ radian} = 180/\pi^\circ$ , then we can get a very good fit indeed! [f\\_130-131](#)

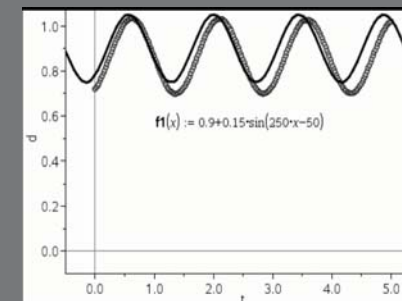
Of course there should be decay over time, until eventually the elephant is at rest! We just need to log more oscillations to see this. Another interesting view is to divide a page up into several different Data & Statistics graphs. Here the distance, velocity and acceleration graphs against time are showed separately, together with a different type of representation in the bottom right. This is effectively a parametric plot which winds round according to the time  $t$ . If there was no loss of energy the curve would be closed (probably an ellipse?) – but otherwise it would actually be an inward winding spiral. It is known as the **phase-plane portrait** of the spring-mass system. When the velocity is positive it is moving from right to left, and vice versa. [f\\_132](#)



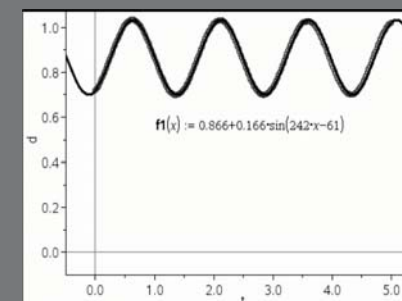
[f\\_125](#)

t	d	v	a
dc01.time	dc01.dist1	dc01.vel1	dc01.acc1
0	0.719572	0.358433	-0.935351
0.02	0.72695	0.351305	1.03089
0.04	0.733072	0.402271	3.88138
0.06	0.742307	0.523583	4.21898
0.08	0.754689	0.583479	1.29617
0.1	0.766142	0.584621	0.061855
0.12	0.777767	0.604979	-0.289273
0.14	0.790304	0.63615	0.063747
0.16	0.803236	0.665379	0.550318
0.18	0.816942	0.688237	0.982779

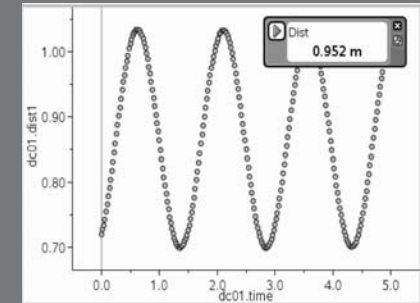
[f\\_127](#)



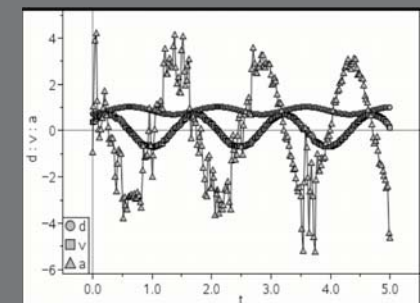
[f\\_129](#)



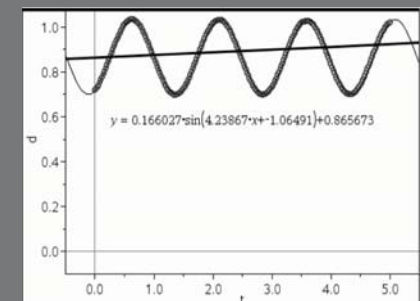
[f\\_131](#)



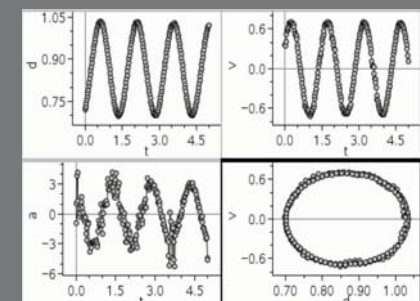
[f\\_126](#)



[f\\_128](#)



[f\\_130](#)



[f\\_132](#)



## Limits to growth - falling from a plane

At the end of September, 1989 there was a story in the newspapers of a very fortunate young lady who survived a fall of 3,500 ft from an aeroplane when her parachute failed to open. It also mentioned two other previous instances of people surviving such falls, one from 18,000 ft and one from 33,300 ft (the world record still!).

*"JAT stewardess Vesna Vulović survived a fall of 33,000 feet (over 10,000 meters) on January 26, 1972 when she was thrown from JAT Flight 364. The plane was brought down by explosives planted by Croatian (Ustashe) terrorists, over Srbská Kamenice in the former Czechoslovakia (now Czech Republic). The Serbian stewardess suffered a broken skull, three broken vertebrae, one crushed completely, and was in a coma for 27 days."*



The concluding paragraph gave the following useful information: "it is estimated that the body usually reaches 99% of its low-level terminal velocity after falling 1,880 ft which takes about 14 seconds. The speed of a falling body varies from 117 to 125 mph in random posture, but can be accelerated to 185 mph in a head down position".

Taking 1 ft as 0.305 m and 1 mph as  $0.447 \text{ ms}^{-1}$  we can see if it is possible from this data to construct a reasonable model of resisted 1-dimensional motion under gravity (that of Galileo). Working to round figures we can take the 3 falls mentioned as from heights of 1,000 m, 5,000 m and 10,000 m, and the terminal velocity as  $53 \text{ ms}^{-1}$ , with 99 per cent, of terminal velocity reached after 570 m. First can you find out what speeds the three unfortunates would have reached if there had been no air resistance? Use the formula  $v^2 = u^2 + 2as$  where  $v$  is the final velocity,  $u$  is the initial velocity,  $s$  is the distance travelled and  $a$  is the acceleration (assumed constant) and the assumptions that vertically  $u=0$  and  $a=g \approx 9.8 \text{ ms}^{-2}$ , and hence find the three final velocities in  $\text{ms}^{-1}$  and mph. Would any or all have broken the sound barrier?

The usual model for air resistance  $R$  is to assume that  $R$  is proportional to some power of the velocity:  $R = k.v^p$  where the constant  $k$  depends upon the "stickiness" of the air, the profile of the body etc., and where the power  $p$  (called the **exponent**) depends upon what sort of range of speeds are involved. The equation of motion is  $F = m.a$ , and we assume the mass is not changing!

Letting  $k = m.c$  we have  $a = g - c.v^p$ . Suppose we start with a simple **linear** model:  $p = 1$ . Then the body will have reached terminal velocity when  $a = 0$ , i.e.  $c = g/v = 9.81/53 = 0.185$ . So the acceleration becomes  $a = 9.81 - 0.185 v$ . Since the acceleration is the rate of change of the velocity this can be written as a differential equation:  $dv/dt = 9.81 - 0.185 v$  - which requires a knowledge of calculus to solve. Instead we will set up a spreadsheet and copy down formulae for velocities and distances using a finite time step  $dt$ . The column B is used to define the constants used, C holds time  $t$ , D holds velocity  $v$  and E holds distance  $s$ . The recurrence formulae which are copied are:

$$c1 = c0 + dt \quad d1 = d0 + (g - c \cdot v^p) \cdot dt \quad e1 = e0 + ((d0 + d1)/2) \cdot dt$$

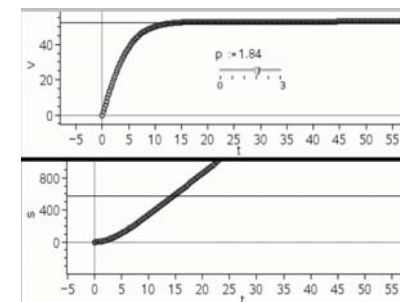
We scan the E column looking for a close match to 570m. In row 80 we find that  $t = 15.8 \text{ secs}$ ,  $v = 50.3 \text{ ms}^{-1}$  and  $s = 570.6 \text{ m}$ . The velocity should be 99% of the terminal velocity i.e.  $52.47 \text{ ms}^{-1}$ . So now we need to start to experiment with the power  $p$ . [f\\_133](#)

We can illustrate this graphically by comparing data plots of the velocity and distance graphs against  $t$ . We need to find a value of  $p$  so that the lines intersect the graphs at the same value of  $t$ . [f\\_134](#)

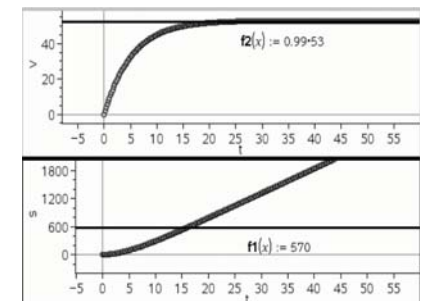
We find with  $p = 2$  that that  $t = 14.4 \text{ secs}$ ,  $v = 52.6 \text{ ms}^{-1}$  and  $s = 568.7 \text{ m}$ . So that is a closer fit and suggests we need a value of  $p$  between 1 and 2, but nearer 2. With  $p = 1.84$  we see a good fit. [f\\_135](#)

	A	B	C=t	D=v	E=s	F	G
1	g	9.81	0	0	0		
2	vt	53	0.2	1.962	0.1962		
3	c	0.185094	0.4	3.85137	0.77537		
4	dt	0.2	0.6	5.6708	1.72975		
5	p	1	0.8	7.42287	3.03912		
6			1	9.11008	4.69242		
7			1.2	10.7348	6.67691		
8			1.4	12.2994	8.98034		
9			1.6	13.8061	11.5909		
10			1.8	15.257	14.4972		

[f\\_133](#)



[f\\_135](#)



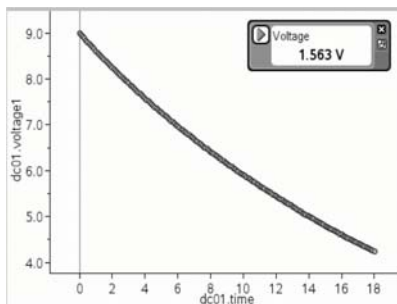
[f\\_134](#)

We find with  $p = 1.84$  that that  $t = 14.6$  secs,  $v = 52.43 \text{ ms}^{-1}$  and  $s = 573.3 \text{ m}$ . So we found a model of air-resistance proportional to the 1.84 power of the velocity which fits well with the data given in the article. In reality it appears that the power itself also depends upon the velocity – with roughly a linear behaviour for small velocities, quadratic for medium velocities etc. The whole field of resisted motion and ballistics is very much a non-exact science!

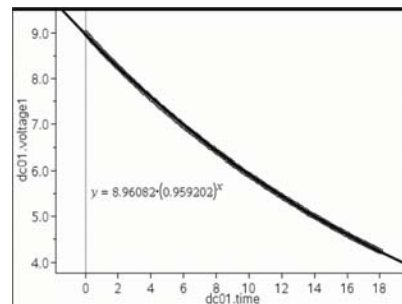
## Growth and decay

In order to produce a simple model for studying decay you just need a few inexpensive electronic components. We use a voltage probe together with Vernier's Go!Link to plot the discharge of a  $220 \mu\text{F}$  capacitor wired in parallel with a  $100 \text{ k}\Omega$  resistor across a  $9\text{V}$  cell. The cell is disconnected as the data-capture starts. So we see a very nice decay curve which will eventually settle to a voltage of zero when the capacitor is fully discharged. *f\_136*

This time we can fit a different regression model, called an Exponential regression. Here the word “exponent” just means a power. We see that a very good fit is given by the function  $y = 8.96 \cdot 0.96^x$  *f\_137*



*f\_136*

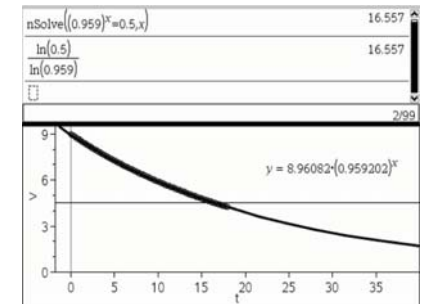


*f\_137*

We can study this more closely in a Lists & Spreadsheets page. The list of times is entered into the first column as the variable  $t$ , and of voltages into the second column as the variable  $v$ . The third column holds the ratios of successive values of  $v$  e.g.  $c1 = b2/b1$  and this is copied right down until  $t = 17.9 \text{ s}$ . These ratios are pretty much all equal – representing a decay of about 0.6% every 1/10 of a second. The cell d1 holds the mean of these ratios. As this is the ratio per 0.1 of a second, the ratio per second is given by the 10<sup>th</sup> power of this – which agrees with the value found in the exponential regression. At time  $t=0$  the initial voltage was 8.998 – so our “best guess” at a good fit would have been  $v = 8.998 (0.959)^t$ . *f\_138*

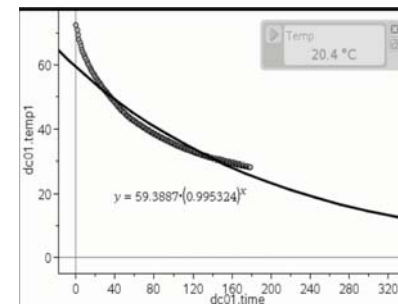
t	v	r			
=dc01.time	=dc01.voltage1				
1	0.	8.9978	0.995557	0.995825	mean(r)
2	0.1	8.95782	0.996048	0.959029	d1^10
3	0.2	8.92242	0.995519		
4	0.3	8.88245	0.995465		
5	0.4	8.84216	0.995444		
6	0.5	8.80188	0.995423		
7	0.6	8.7616	0.995994		
8	0.7	8.7265	0.995384		
9	0.8	8.68622	0.99596		
10	0.9	8.65112	0.995344		
11		8.61094	0.995934		

*f\_138*



*f\_139*

Another similar form of decay model applies to radioactivity, where an important measure is called the half-life. This is the length of time required for the radiation strength to decay by 50%. We can compute this for our simple condenser. We just need to solve  $(0.959)^t = 0.5$ . You can do this on a Calculator screen. One means to solve such an equation involves the use of logarithms, which were invented by the Scottish mathematician John Napier (1550-1617). Here we can see that the half-life of the condenser/resistor circuit is about 16.6 sec – and you could look up something about so-called RC-circuits to check whether this agrees with the physics theory! *f\_139*



*f\_140*

An important use of radioactive decay is in dating archaeological finds by a process known as **Carbon dating**, using the measurement of radioactivity of Carbon 14. See if you can find some information on the process and produce a TI-Nspire notebook to present your results.

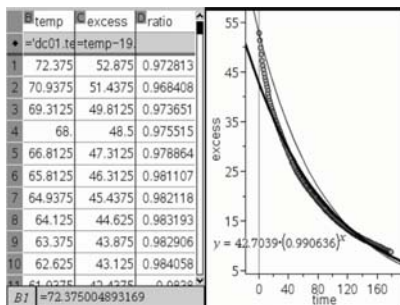
Another famous model was produced by Isaac Newton to predict the cooling of a substance. In this case we will use a temperature probe and a cup of hot water to produce a cooling curve. This time we use a Go!Temp probe and place it into a mug of very hot water. Then we take it out to cool in air and at the same time start the data-logging from a Data & Statistics page. *f\_140*



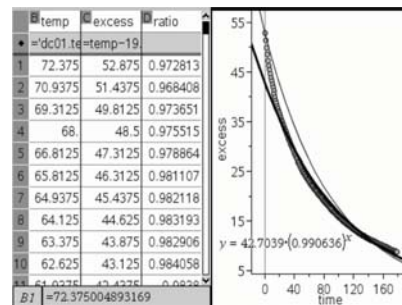
If we adjust the scales to see more of the graph, we can see that fitting an exponential regression model does not work well. The main reason is that as  $x$  gets larger and larger the value of  $y$  tends to zero. Which in our case would mean the temperature probe's tip would end up freezing! *f\_141*

Newton introduced the idea of the prevailing temperature of the environment. For example if the data collection took place in a room whose temperature was regulated to be  $19.5^\circ$  then the long term behaviour of the probe would be that it cooled towards room temperature – so we should look at modelling the excess of the temperature over the environment. *f\_142*

We can set up sliders to explore how well the function  $y = a.bx + c$  can be made to fit the cooling data. It is from this kind of model that forensic scientists can determine the cause of death from the blood temperature of a corpse!



f\_141



f\_142

# Conclusion

In this exploration of modelling techniques we have visited some of the most important physical models. We haven't examined bouncing balls, pendulums, motion in a circle, friction and many other aspects of rich sources of data. We hope that you might feel the urge to find out more about some of these aspects and see if you can apply some of the modelling skills we have illustrated here.

Another rich source of models comes from the life sciences, and including population modelling (also known as demography), drug ingestion and the spread of infection. Again you might like to find out more about these and see if you can apply your skills to them. Names to look out for include Malthus, Verhulst, Lotke, Volterra, Lanchester, Fry Richardson and topics include predator-prey, logistic growth, host-parasite and the SIR model. Happy hunting!