Name _____

Class _____

Problem 1 – A secret message

Matrices are often used to encode secret messages. For example, using the chart to the right, the word

SYSTEM

can be written using numerals as

19 25 19 20 5 13

and then recorded in a matrix as

$$\begin{bmatrix} 19 & 25 \\ 19 & 20 \\ 5 & 13 \end{bmatrix}.$$

_ = 0	**I** = 9	**R** = 18
A = 1	**J** = 10	**S** = 19
B = 2	**K** = 11	**T** = 20
C = 3	**L** = 12	**U** = 21
D = 4	**M** = 13	**V** = 22
E = 5	**N** = 14	**W** = 23
F = 6	**O** = 15	**X** = 24
G = 7	**P** = 16	**Y** = 25
H = 8	**Q** = 17	**Z** = 26

To protect this message as it is transmitted, it is *encoded* by multiplying the message matrix by an encoding matrix, such as $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 19 & 25 \\ 19 & 20 \\ 5 & 13 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 126 & 107 \\ 116 & 97 \\ 46 & 41 \end{bmatrix}$$

So the message becomes **126 107 116 97 46 41**.

The receiver of this message can retrieve the original message by *decoding* it using the *inverse* of the coding matrix.

The **inverse** of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, provided $\det(A) \neq 0$, is $A^{-1} = \dfrac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

- What is our decoding matrix, the inverse of our encoding matrix?

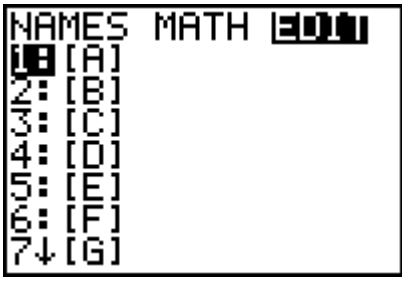
  (Recall that $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.)

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}^{-1} = \frac{1}{\square} \cdot \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

- Test the decoding matrix to verify that these matrices are inverses. Show your work. (Multiplying a matrix by its inverse—in either direction—should result in the identity matrix. Follow the steps below to use the calculator.)

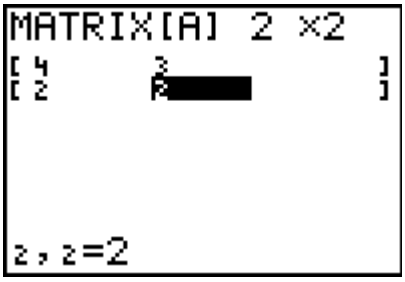
To do the multiplication, you first have to set up the matrices. Press [2nd] [MATRIX] to access the matrix menu.

Arrow to **EDIT** and press [ENTER].

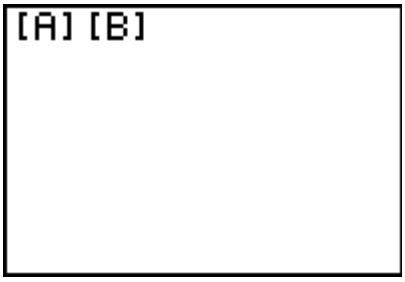


Make matrix [A] a 2 × 2 by changing the numbers in the top right corner of the screen and then enter in the numbers for the *encoding* matrix.

Make matrix [B] also 2 × 2 and enter in the numbers for the *decoding* matrix.



Once the matrices are created, go to the home screen by pressing [2nd] [QUIT]. Now multiply the encoding matrix and the decoding matrix to confirm that they are inverses.



- Now test the decoding matrix on our encoded message and check against the chart. Show your work.

Problem 2 – Solving systems of equations

You can use inverse matrices to help you solve a system of equations.

First, write the system as a matrix equation of the form $AX = B$, where *A* is the coefficient matrix, *X* is the variable matrix, and *B* is the constant matrix.

$$AX = B$$

Then, you just need to multiply both sides of the equation by the inverse of the coefficient matrix, as shown here. (Be careful with the order in which you multiply the matrices!)

$$A^{-1} \cdot AX = A^{-1} \cdot B$$
$$X = A^{-1} \cdot B$$

Let's try an example.

- Write this system as a matrix equation.

$$\begin{aligned} x - 2y + z &= 7 \\ 3x - 5y + z &= 14 \\ 2x - 2y - z &= 3 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

- Multiply the inverse of the coefficient matrix by the constant matrix.
  (Use the Matrix menu to set up [A] as a 3 × 3 matrix and [B] as a 3 × 1 matrix.)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

- Check your solutions back in the original system of equations. Show your work.

Problem 3 – Finding the equation of a parabola

How do you think you could use inverse matrices to find the equation of a parabola that passes through the points (–1, 5), (2, –1), and (3, 13)?

Recall that parabolas are of the form $y = ax^2 + bx + c$.

- Replace the *x*s and *y*s with the given coordinates to obtain a system of three equations.
- Rewrite this as a matrix equation and solve as in Problem 2.
- Substitute the resulting values of *a*, *b*, and *c* into the quadratic equation and check that the points indeed satisfy the equation.

Use your calculator to perform your calculations.

- **<u>System of equations</u>:**

- **<u>Matrix equation</u>:**

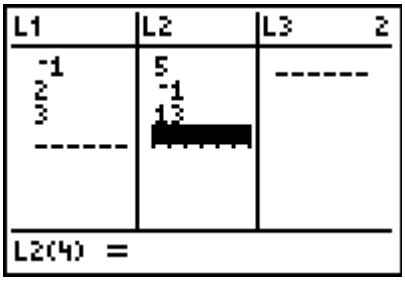
$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

- **<u>Solution</u>:**

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

- **<u>Equation of parabola</u>:**

Check that the points satisfy your resulting equation. Press [STAT] [ENTER] and enter the *x*-values into **L1** and the *y*-values into **L2**.



Press [2nd] [STAT PLOT] and turn on **Plot1** with L1 as the Xlist and L2 as the Ylist.

Enter the equation of the parabola in **Y1**. Set an appropriate window and then press [GRAPH].

Exercises

1. Decode the message **18 27 51 81 37 58 60 100 18 27 85 137 59 93 51 79**, which was encoded with encoding matrix $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

(Remember to use the chart on the first page to decipher the results.)

2. Solve the system below using inverse matrices.

$$\begin{aligned} 7x - 8y + 5z &= 18 \\ -4x + 5y - 3z &= -11 \\ x - y + z &= 1 \end{aligned}$$

3. Find the equation of the parabola that passes through the points (–1, 3), (1, –3), (2, 0).