Proof of Identities

MATH NSPIRED

Math Objectives

- Students will be able to interpret reciprocal, negative angle, cofunction, and Pythagorean identities in terms of the graphs of the trigonometric functions involved.
- Students will be able to prove trigonometric identities algebraically.
- Students will be able to visualize trigonometric identities graphically.
- Students will make conjectures and build a logical progression of ideas (CCSS Mathematical Practice).
- Students will use technology to visualize the results of varying assumptions (CCSS Mathematical Practice).
- Students will use technological tools to explore and deepen understanding of concepts (CCSS Mathematical Practice).

Vocabulary

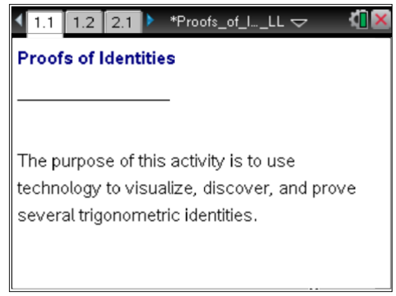
- Cofunction Identities
- Identity
- Negative Angle Identities
- Pythagorean Identities
- Reciprocal Identities
- Trigonometric Identities

About the Lesson

- This lesson involves discovering, visualizing, and proving trigonometric identities.
- As a result, students will:
  - Manipulate the graphs of trigonometric functions.
  - Utilize sliders to discover and support trigonometric identities.
  - Drag a point to see its relationship to its reflected image and use this information to discover the Negative Angle Identities.
  - Utilize the relationship between an angle and its complement to discover the Cofunction Identities.

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to examine patterns that emerge and monitor students' understanding.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Files:

Student Activity

Proofs_of_Identities_Student.pdf

Proofs_of_Identities_Student.doc

TI-Nspire document

Proofs_of_Identities.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

- Use Live Presenter to share students' ideas.
- Use Quick Poll to assess students’ understanding.

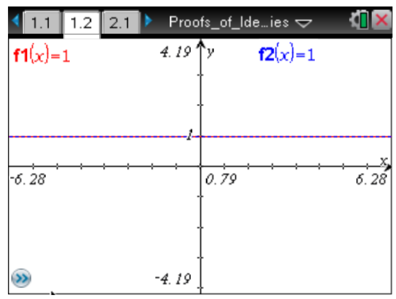
Discussion Points and Possible Answers

Tech Tip: If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand ( ) getting ready to grab the point. Then press ctrl [click] to grab the point and close the hand ( ). When finished moving the point, press esc to release the point.

Teacher Tip: You might want to use this activity in its entirety, or pick and choose various components to use at appropriate times in the curriculum. Both the .doc and .tns files can be edited as needed.

Move to page 1.2.

This page shows the graphs of two identical functions: f1(x) = 1 and f2(x) = 1. Throughout this activity, the graph of f1(x) is shown as a thin, solid line and the graph of f2(x) is shown as a thicker dashed line.



1. Press ctrl G to access the function entry line. Press the up arrow (▲) to access the equation for f1(x) and change it to csc(x). Change the equation for f2(x) to $\dfrac{1}{\sin(x)}$.

   What do you notice about the two graphs? Why is your conclusion true?

   Answer: The two graphs seem to be identical. We know that the two graphs actually are identical because $\csc(x)$ is defined as $\dfrac{1}{\sin(x)}$.

2. Complete the information for the other "Reciprocal Identities". To do so, you might want to change the equations for f1(x) and f2(x) to help you visualize the identities.

   a. $\sec(x) =$

   Answer: $\dfrac{1}{\cos(x)}$.

   b. $\cot(x) =$

Answer: $\frac{1}{\tan(x)}$.

c. $\sin(x) =$

Answer: $\frac{1}{\csc(x)}$.

d. $\cos(x) =$

Answer: $\frac{1}{\sec(x)}$.

e. $\tan(x) =$

Answer: $\frac{1}{\cot(x)}$.

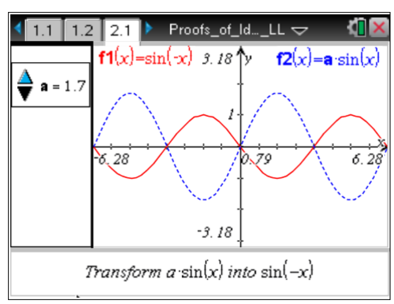
TI-Nspire Navigator Opportunity: ***Live Presenter and Quick Poll***
See Note 1 at the end of this lesson.

3. Why are these statements called Reciprocal Identities?

Answer: These statements are called Reciprocal Identities because the product of each pair of functions is 1. For example, $\cos(x)\cdot\sec(x) = \cos(x)\cdot\frac{1}{\cos(x)} = 1.$

Teacher Tip: Be sure to remind students that, if cos(x) = 0, sec(x) will be undefined and their product is not 1. You may want to spend a few minutes discussing the domain of the given functions.

Move to page 2.1.



Click on the slider to transform $f2(x) = a\cdot\sin(x)$ into $f1(x) = \sin(-x)$.

4. How does changing the parameter, *a*, affect the graph of $f(x) = a\cdot\sin x$?

Answer: The value of *a* affects the height of the sine curve. If $|a| > 1$, the graph is stretched vertically. If $0 < |a| < 1$, the graph is vertically compressed. If $a < 0$, the graph is reflected over the x-axis.

Teacher Tip: If students are not familiar with the term amplitude, you might want to take a moment to introduce it. We say that $|a|$ is the amplitude, which is the vertical distance between the function's maximum (or minimum) value and the midline (in this case, the x-axis).

5. Write the equation for $f2(x)$ with a specific value for *a* such that $f2(x) = f1(x)$.

Answer: $a = -1$ and $f2(x) = -\sin(x)$.

TI-Nspire Navigator Opportunity: *Screen Capture and Quick Poll*
See Note 2 at the end of this lesson.

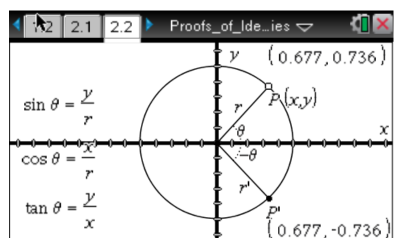
6. Using the information above, fill in the blank:

$\sin(-x) = \_\_\_\_\_ \sin(x)$.

Answer: $\sin(-x) =$ -1 $\sin(x)$.

Move to page 2.2.

This page shows the unit circle and the definitions for three basic trigonometric functions in terms of *x*, *y,* and *r*. The segment labeled *r* has been reflected over the *x*-axis. Drag point *P,* and observe how the coordinates of point P and its image point P' change.



Proof of the negative angle identities

7. Consider the relationship between the coordinates of P and the coordinates of P'. Write the coordinates of P and the image point, P', in terms of *x* and *y*.

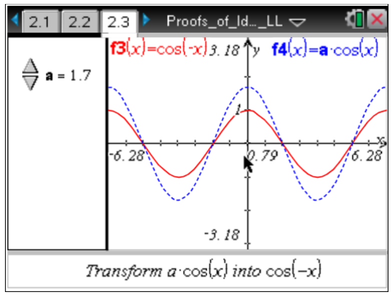
Answer: The *x*-coordinate of the image point remains the same. The *y*-coordinate of the image point is the negation of the *y*-coordinate of the original point. Therefore, for every original point $P(x, y)$, the coordinates of the image point are $P'(x, -y)$.

8. Using the coordinates of the image point P' and the definitions of the basic trigonometric functions shown on the screen, write an expression for sin($-\theta$) in terms of sin(θ).

Answer: In a unit circle, the sine of a central angle is equal to the *y*-coordinate of the point where the terminal side of the angle intersects the unit circle. Thus, $\sin(\theta) = y$ and $\sin(-\theta) = -y$. Therefore, $\sin(-\theta) = -\sin(\theta)$.

Teacher Tip: You might want to make the connection between this concept and the findings in questions 5 and 6 above.

Move to page 2.3.



Click on the slider to transform $f4(x) = a \cdot \cos(x)$ into $f3(x) = \cos(-x)$.

9. How does changing the parameter, *a*, affect the graph of $f(x) = a \cdot \cos x$?

Answer: The value of *a* affects the height of the cosine curve. If $|a| > 1$, the graph is stretched vertically. If $0 < |a| < 1$, the graph is vertically compressed. If $a < 0$, the graph is reflected over the *x*-axis.

10. Follow the directions in questions 5 and 6, using the cosine function instead of the sine function. Write the equation for $f4(x)$ in the space below.

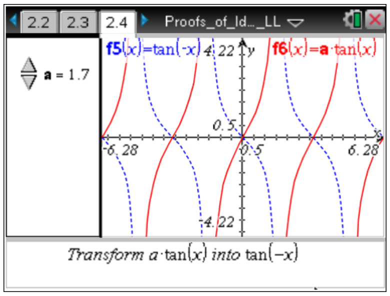
Answer: $a = 1$ and $f4(x) = \cos(x)$.

Move back to page 2.2.

11. Using the coordinates of the image point P' and the definitions of the basic trigonometric functions shown on the screen, write an expression for cos($-\theta$) in terms of cos(θ).

Answer: In a unit circle, the cosine of a central angle is equal to the *x*-coordinate of the point on the circle. Thus, $\cos(\theta) = x$ and $\cos(-\theta) = x$. Therefore, $\cos(-\theta) = \cos(\theta)$.

Move to page 2.4.



Click on the slider to transform $f6(x) = a \cdot \tan(x)$ into $f5(x) = \tan(-x)$.

12. Follow the directions in questions 5 and 6, using the tangent function instead of the sine function. Write the equation for $f6(x)$ in the space below.

Answer: $a = -1$ and $f6(x) = -\tan(x)$.

13. Use the information you obtained about $\sin(-\theta)$ and $\cos(-\theta)$ in questions 8 and 11 to write an expression for tan(–θ) in terms of tan(θ).

Answer: The tangent function is defined by $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$. Therefore,

$\tan(-\theta) = \dfrac{\sin(-\theta)}{\cos(-\theta)} = \dfrac{-\sin(\theta)}{\cos(\theta)} = -\dfrac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$. Thus, $\tan(-\theta) = -\tan(\theta)$.

> **Teacher Tip:** You might want to ask students how this information connects with the information taken from the unit circle.

14. Use the information from above to fill in the information below for the "Negative Angle Identities."

a. $\sin(-\theta) =$

Answer: $-\sin(\theta)$.

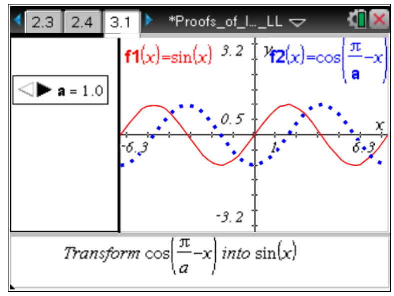
b. $\cos(-\theta) =$

Answer: $\cos(\theta)$.

c. $\tan(-\theta) =$

Answer: $-\tan(\theta)$.

> **Teacher Tip:** If students have learned about the concept of odd and even functions, you might want to make the connection between this concept and the Negative Angle Identities.



Move to page 3.1.

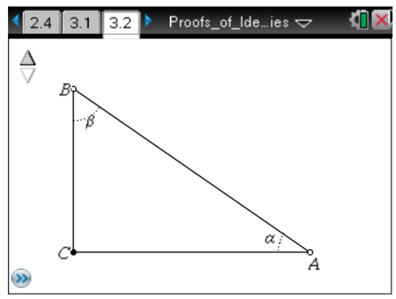
Click on the slider to transform $f2(x)=\cos\left(\frac{\pi}{a}-x\right)$ into $f1(x)=\sin(x)$.

15. How does changing the parameter, *a*, affect the graph of $f(x)=\cos\left(\frac{\pi}{a}-x\right)$?

Answer: Changing the value of *a* results in a horizontal shift of the graph.

16. Use the equation for $f2(x)$ to fill in the blank:

Answer: $\sin(x)=\cos\left(\frac{\pi}{2}-x\right)$.



Move to page 3.2.

This page shows a right triangle with acute angles α and β labeled. Press the arrow (▲) to see the image of the triangle after it is reflected over its hypotenuse. Press the arrow (▲) again to see the measures of two additional angles labeled as α and β.

Teacher Tip: Students might already be quite comfortable with the fact that α and β are complementary. If so, you might want to skip some of the more basic questions below.

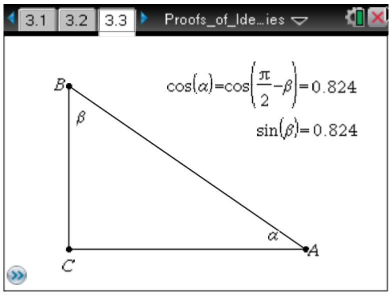
17. Why are the two angles in the image labeled as they are?

Answer: When the right triangle is reflected over its hypotenuse, the two triangles form a rectangle. Since both pairs of opposite sides of a rectangle are parallel, transversal $\overline{AB}$ forms two pairs of congruent alternate interior angles.

18. What is the relationship between α and β? Explain your answer.

Answer: The two angles form a right angle, so α and β are complementary.

Move to pages 3.3.



19. a. Click and drag vertex *A* and vertex *B*. What do you notice about $\sin(\beta)$ and $\cos(\alpha)$? Write an equation to express $\sin(\beta)$ in terms of α.

<u>Answer:</u> We see that $\sin\beta = \cos\alpha$.

b. Substitute $\frac{\pi}{2}-\beta$ for α to re-write the equation above in terms of β.

<u>Answer:</u> We see that $\sin\beta = \cos\alpha = \cos\left(\frac{\pi}{2}-\beta\right)$. Thus, $\sin\beta = \cos\left(\frac{\pi}{2}-\beta\right)$.

20. We see from questions 17 - 19 that the sine of an acute angle is equal to the cosine of its complement. The sine and cosine functions are called *cofunctions*. The tangent and cotangent functions are also *cofunctions*. Use this information to fill in the blanks below for the "Cofunction Identities."

a. $\sin(\theta) = \cos($ _____________ $)$

<u>Answer:</u> $\sin(\theta) = \cos\left(\frac{\pi}{2}-\theta\right)$.

b. $\cos(\theta) = \sin($ _____________ $)$

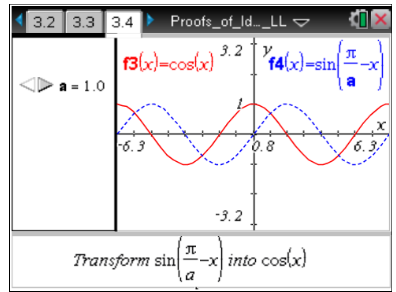
<u>Answer:</u> $\cos(\theta) = \sin\left(\frac{\pi}{2}-\theta\right)$.

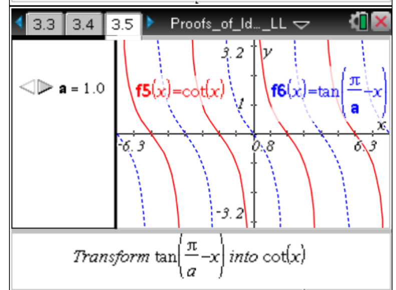
c. $\tan(\theta) =$ ___________________

<u>Answer:</u> $\tan(\theta) = \cot\left(\frac{\pi}{2}-\theta\right)$.

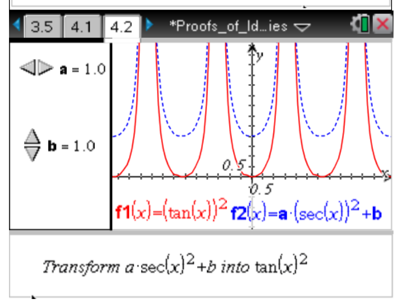
Move to pages 3.4 and 3.5.

Use these pages to support the identities you wrote in question 20.





Move to page 4.2.

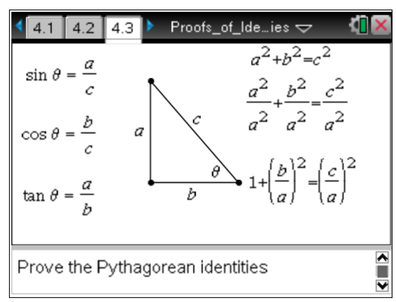


Move the sliders to change the values of the two parameters in $f2(x)$ to transform the graph of $f2(x)=a\cdot(\sec(x))^2+b$ into $f1(x)=(\tan(x))^2$.

21. Write the equation for $f2(x)$ with specific values for *a* and *b* such that $f2(x)=f1(x)$.

<u>Answer:</u> For a = 1 and b = -1, $f2(x)=(\sec(x))^2-1$.

Move to page 4.3.



This page shows a right triangle and the definitions of three basic trigonometric functions in terms of the lengths of the sides.

22. Divide both sides of the Pythagorean Formula by a^2 as shown, and simplify the result. Substitute the appropriate trigonometric functions for the ratios $\frac{b}{a}$ and $\frac{c}{a}$, and rewrite the equation below.

<u>Answer:</u> We see that $1+\left(\frac{b}{a}\right)^2=\left(\frac{c}{a}\right)^2$. Since $\frac{b}{a}=\cot\theta$ and $\frac{c}{a}=\csc\theta$, we know that $\left(\frac{b}{a}\right)^2=\cot^2\theta$ and $\left(\frac{c}{a}\right)^2=\csc^2\theta$. Substituting this information into the equation, we get

$1+\cot^2\theta=\csc^2\theta$.

23. Repeat the process, dividing by b^2 and then by c^2 to yield two additional identities. Fill in the information below for the "Pythagorean Identities."

a. ____________ = $\sec^2(x)$

Answer: $\tan^2(x) + 1$.

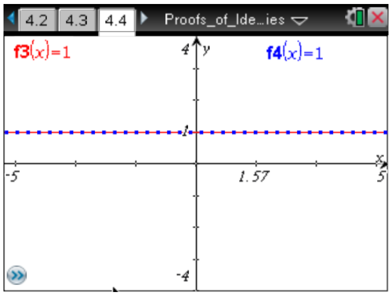
b. ________+ $\cot^2(x)$ = _________

Answer: $1 + \cot^2(x) = \csc^2(x)$.

c. $\sin^2(x)$ + _______= ________

Answer: $\sin^2(x) + \cos^2(x) = 1$.

Move to page 4.4.



Use this page to visually support the identities you derived in question 23. Press ctrl G to access the function entry line. Press the up arrow (▲) to access f3(x). Enter the left half of the equation from #23b as f3(x) and the right half of the equation as f4(x). Repeat the process for the equation for #23c.

24. a. Using the Pythagorean Identity, $\sin^2\theta+\cos^2\theta=1$, divide all three terms by $\sin^2\theta$ and simplify the fractions to write another Pythagorean Identity.

Answer: Given that $\sin^2\theta+\cos^2\theta=1$, we divide all three terms by $\sin^2\theta$ as shown below:

$$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

$$1+\left(\frac{\cos\theta}{\sin\theta}\right)^2=\left(\frac{1}{\sin\theta}\right)^2$$

$$1+\cot^2\theta=\csc^2\theta$$

b. Use the identity $\sin^2\theta+\cos^2\theta=1$ to divide all three terms by $\cos^2\theta$, and simplify the fractions to write another Pythagorean Identity.

Answer: Given that $\sin^2\theta+\cos^2\theta=1$, we divide all three terms by $\cos^2\theta$ as shown below:

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2+1=\left(\frac{1}{\cos\theta}\right)^2$$

$$\tan^2\theta+1=\sec^2\theta$$

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- That an identity is true for all values for which the variable is defined.
- How to manipulate the graphs of trigonometric functions to discover trigonometric identities.
- That although a graph can be used to support the validity of a trigonometric identity, it does not constitute a proof.

Assessment

You might want to combine several identities into one expression and ask students to simplify the expression. For example, ask students to rewrite the expression $\dfrac{\sin^2 x+\cos^2 x}{\cos x}$ as a single trigonometric function.

TI-Nspire Navigator

Note 1

Name of Feature: Live Presenter and Quick Poll

Ask one of the students to serve as the Live Presenter and demonstrate to the class how to navigate through the activity. Use a Quick Poll to collect students' answers and assess their understanding.

Note 2

Name of Feature: Screen Capture and Quick Poll

A Screen Capture can be used to capture screen shots of the types of functions that students are creating. You might want to leave Screen Capture running in the background, with a 30-second automatic refresh, and without student names displayed. This will enable you to monitor students' progress and make the necessary adjustments to your lesson. You might want to send periodic Quick Polls throughout the lesson to ask students for their equations.