Math Objectives

- Students will be able to use the slope of the symmetric secant line to approximate the derivative of a function at a point and generalize properties of functions that affect the accuracy of these estimates.
- Students will be able to explain the relationship between the symmetric difference quotient and the standard difference quotient used to calculate the derivative of a function at a point both graphically and numerically.

Vocabulary

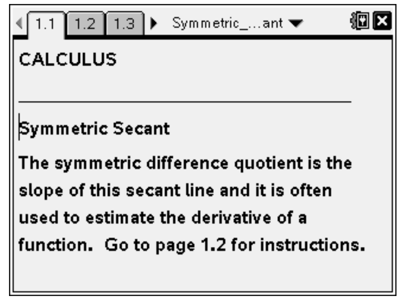
- secant and tangent line
- difference quotient
- derivative

About the Lesson

- This lesson provides a visual demonstration of how and when the symmetric secant line can be used to provide a reasonable estimate for the derivative of a function at a point.
- As a result, students will:
  - Explore a variety of function graphs to observe how the slope of the symmetric secant line comes closer to approximating the slope of the tangent line as the value of *h* decreases.
  - Use the symmetric secant line to estimate derivatives at a point and compare these estimates to other numerical and analytic methods.
  - Discover the importance of considering the function graph when estimating derivatives by exploring instances in which the symmetric difference quotient provides a value even though the derivative of the function does not exist.

TI-Nspire™ Navigator™ System

- Use Screen Capture to check students' understanding and to explore a wider variety of examples.
- Use Quick Poll to assess students' understanding of connections between numerical and graphical methods for estimating the derivative.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point
- Click a minimized slider

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Materials:

Student Activity
Symmetric_Secant_Student.pdf
Symmetric_Secant_Student.doc

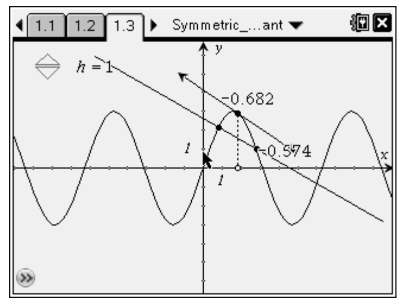
TI-Nspire document
Symmetric_Secant.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

Tech Tip: If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand (☝) getting ready to grab the point. Also, be sure that the word *point* appears, not the word *text.* Then press ctrl ☝ to grab the point and close the hand (✊).

Open the TI-Nspire document *Symmetric_Secant.tns.*



The Symmetric Difference Quotient given by: $\frac{\mathbf{f}(x+h)-\mathbf{f}(x-h)}{2h}$ is often used to approximate the derivative of a function $\mathbf{f}(x)$ at a point. In this activity, you will explore the symmetric difference quotient both graphically and numerically to consider its benefits and limitations.

Teacher Tip: Before beginning this activity, students should be familiar with the definition of a derivative at a point as the slope of the tangent line at that point.

Move to page 1.2 for instructions and then to page 1.3.

1. Page 1.3 shows the graph of $y = \mathbf{f}(x)$ and the tangent line through the point (x, $\mathbf{f}(x)$). The slope of the secant line connecting the points (x – *h*, **f1**(*x* – *h*)), and (*x* + *h*, **f1**(*x* + *h*)) is also given.
   a. Explain why the slope of the secant line is represented by the Symmetric Difference Quotient given above.

   <u>Sample answer:</u> The symmetric difference quotient is the same as the slope of the line calculated using the ordered pairs: slope of secant = $\frac{\Delta y}{\Delta x} = \frac{\mathbf{f}(x+h)-\mathbf{f}(x-h)}{(x+h)-(x-h)}$, which simplifies to $\frac{\mathbf{f}(x+h)-\mathbf{f}(x-h)}{2h}$.

   Teacher Tip: Students should also note why this line is appropriately called the "symmetric" secant because you are approximating the slope at the point (x, $\mathbf{f}(x)$) by using x-values the same small distance, h, on either side.

b. Drag the point on the *x*-axis to change the *x*-value. Describe the changes in the secant and tangent lines as you move along the graph of *y* = **f1**(*x*).

Sample answer: At times, the lines are close to parallel, with the secant line following the curve in a similar manner as the tangent line so the slope of the secant line gives a reasonable estimate for the derivative. The slopes of the secant and tangent lines are closer together (i.e., the lines appear closer to parallel) when *x* is near the relative minimum and maximum points. When *x* is near steeper sections of the graph, the slope of the secant line is generally lower than that of the tangent line.

c. Does the slope of the secant line provide an estimate for the derivative of this function? Explain.

Sample answer: Yes. The slope of the secant line is an estimate of the slope of the tangent line at (*x*, **f**(*x*)), which is the same as the derivative **f'**(*x*) of the function.

> **Teacher Tip:** You may wish to discuss what might be meant by a "good" estimate. In this case it is difficult to judge because you are not given any context for the situation, nor are you given the units for *x* and **f**(*x*). In general, students should note the relative values of *x* and *h* and consider when being accurate to the nearest tenth or thousandth may or may not be "good enough."

2. Use the arrows in the upper left corner to change the value of *h*.

a. Describe changes you note to the secant line as you decrease the value of *h*.

Sample answer: As *h* decreases, the symmetric secant line moves closer and closer to the tangent line. When *h* is .05, the slope of the secant line displayed is identical to the slope of the tangent line at almost all points of the graph.

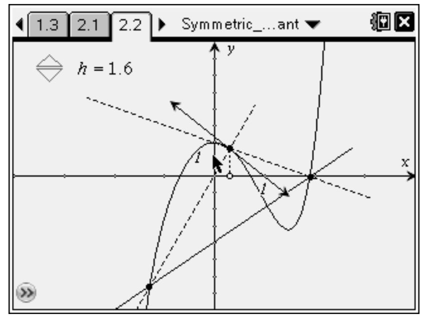
> **Teacher Tip:** Students should repeat this process for a number of different *x*-values. During a class discussion, students should be able to describe and explain why there may be places along the graph where smaller *h* values are needed to give reasonable estimates for the slope of the tangent line.

TI-Nspire Navigator Opportunity: ***Screen Capture***
See Note 1 at the end of this lesson.

b. Explain why decreasing the value of *h* improves the estimates for the derivative.

Sample answer: As *h* becomes smaller, you are getting closer and closer to the actual point (*x*, **f**(*x*)) or to the part of the curve where you are trying to determine the slope. Students who have explored the derivative by zooming in on graphs might also connect the idea of *h* getting smaller to the notion of zooming in closer to that portion of the graph.

Move to page 2.2.



3. The definition of a derivative **f'**(*x*) is often given as $\lim_{h \to 0} \frac{\mathbf{f}(x+h)-\mathbf{f}(x)}{h}$.

a. Explain how this difference quotient can also be interpreted as the slope of a secant line. What are the coordinates of the two points used to calculate the slope this secant line?

Sample answer: This difference quotient can be interpreted as the slope of the secant line calculated from the two points with coordinates (*x*, **f**(*x*)) and (*x* + *h*, **f**(*x* + *h*)). This can be shown by simplifying the slope formula: $\frac{\Delta y}{\Delta x} = \frac{\mathbf{f}(x+h)-\mathbf{f}(x)}{(x+h)-x}$

b. The two dotted lines indicate these new secant lines for positive and negative values of *h*. Drag the point on the *x*-axis to change the *x*-value. Compare all three secant lines with the tangent line for *h* = 1. Which seems to provide a better estimate for the derivative of the function at (*x*, **f**(*x*))? Explain.

Sample answer: In general, the symmetric secant appears more parallel to the tangent line. However, along certain sections of the graph where the function is strictly increasing or decreasing, as between *x* = 0.4 and *x* = 0.7, the slope of the traditional secant line is a better approximation. Because the symmetric secant uses values on both sides of the point of interest (*x*, **f1**(*x*)), it provides better estimates where the graph is changing from increasing to decreasing, etc.

c. Use the arrows to change the value of *h*. How do the secant line approximations change as you decrease the value of *h*?

Sample answer: Both secant lines move closer to the tangent line at a point. In this way, both slopes begin to better approximate the derivative of the function. However, the symmetric secant line often converges "faster" (is closer for larger *h*) than the traditional one-sided secant line.

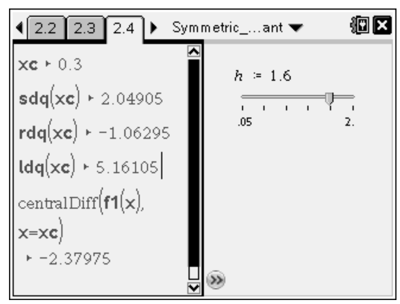
TI-Nspire Navigator Opportunity: ***Screen Capture*** **and** ***Quick Poll***
See Notes 2 and 3 at the end of this lesson.

4. Use the graph to explain why the symmetric difference quotient given by $\frac{\mathbf{f}(x+h)-\mathbf{f}(x-h)}{2h}$ is often a better estimate of the derivative of a function.

   Sample answer: The symmetric difference quotient gives a better estimate because it is taking values from both sides of the *x*-value where you are trying to estimate the slope, and thus provides an average that is closer to the slope at the middle point.

   Teacher Tip: Although the limit of this difference quotient is by definition the derivative of the function at a point, it should be made clear to students that the limit of the symmetric difference quotient as *h* approaches zero is not necessarily the derivative of the function.

Move to page 2.4.



On this page you see numeric values displayed for the symmetric difference quotient, **sdq,** as well as the difference quotients for the traditional secant line from (*x*, **f1**(*x*)) to (*x* + *h*, **f1**(*x* + *h*)). Here **rdq** refers to this quotient when *h* is positive, that is, when the secant line is through a point to the right of (*x*, **f1**(x)), whereas **ldq** is this quotient for negative values of *h*, when the secant line is through a point to the left of (*x*, **f1**(*x*)).

5. Use the slider on this page to decrease the value of *h*. What do you notice about these difference quotients as you decrease the value of *h* toward 0?

   Sample answer: While all of the difference quotients get closer to the numeric derivative as *h* decreases, the symmetric difference quotient more quickly approaches this value. The symmetric difference quotient is always in between the values given by the **rdq** and **ldq.**

6. Use the graph on page 2.2 to change the *x*-value. What is always true about the relationship between these three difference quotients?

   Sample answer: For every *x* value chosen, the symmetric difference quotient is the average of the **rdq** and the **ldq.**

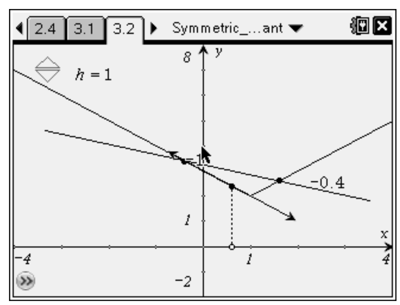
Teacher Tip: Students cannot change the *x*-value at which the quotients are being calculated on page 2.4. They will need to go back to the graph on page 2.2 and drag the point along the *x*-axis as before. The pages are dynamically linked, however, so students will see the new values of **xc** displayed as changes are made. Similarly, using the slider on page 2.4 to change *h* also changes the value of *h* on page 2.2.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 4 at the end of this lesson.

7. How could you use these numeric results to support the claim that the symmetric difference quotient provides a better estimate of the derivative of a function?

   Sample answer: The symmetric difference quotient tends to give a better approximation for the derivative of a function because it is averaging slope values calculated on either side of the point. For many functions, the symmetric difference quotient approaches the slope of the tangent more quickly (for smaller *h* values) than the traditional difference quotient.

Move to page 3.2.



8. Drag the point along the *x*-axis and compare the slope of the tangent line and the slope of the secant line for this new function **f1**(*x*).

   a. For what values of *x* does the slope of the symmetric secant seem to provide a good approximation for the slope of the tangent line?

      Sample answer: The symmetric secant seems to give fairly good approximations. For small enough *h*, the secant line follows the tangent line (and the function itself) exactly for $x < 1$ where the slope is -1 and for $x > 1$ where the slope is 1.

Teacher Tip: Students may not yet note the problem with the function at $x = 1$ and think that the secant line approaching 0 is an acceptable approximation at this point. The next few questions will draw this out for further discussion.

b. Use the symmetric secant to estimate the derivative of the function at:

$x = 3$
Answer: 1

$x = 0$
Answer: –1

$x = 1$
Answer: 0

c. What happens to the tangent line when $x = 1$? What does that tell you about the derivative of the function at $x = 1$?

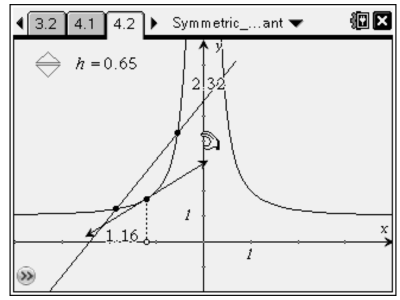
Sample answer: When $x = 1$ the tangent line to the graph actually disappears because the derivative of the function at $x = 1$ does not exist.

> **Teacher Tip:** Watch for students who answer that the tangent line becomes horizontal and the derivative is 0. Clarify why the derivative does not exist at this point either by reviewing the limit definition and noting that the right and left limits are not equal at this point, or by zooming in on the graph at this point to note it will never appear as a straight line.

d. Explain how the symmetric difference quotient might lead to a misrepresentation of the derivative of a function.

Sample answer: Because the symmetric difference quotient is calculating a slope value based only on nearby points, the symmetric difference quotient can provide a value even if the derivative does not exist at that point.

Move to page 4.2.



9. Drag the point along the *x*-axis and compare the slope of the tangent line and the slope of the secant line for this new function **f1**(*x*).

   a. For what values of *x* does the slope of the symmetric secant seem to provide a good approximation for the slope of the tangent line?

      Sample answer: The symmetric secant line appears to give good estimates as long as you are not "too close" to $x = 0$. For $h = 0.05$, the symmetric secant is again very close to the slope of the tangent at (*x,* **f**(*x*)). However, as the *x* values approach zero, the estimates are further from the actual slope of the tangent line. This is often dramatically illustrated as one endpoint of the secant line quickly moves off the screen. Students should also note that when *h* is too large, the secant line might "straddle" the discontinuity.

   b. Use the symmetric secant with $h = 0.05$ to approximate the derivative of this function at:
      $x = –2$
      Answer: 0.25

      $x = 1$
      Answer: –2.01

      $x = 0$
      Answer: Answers may vary. Students may record the last number they see, anything from 32 to –284. Others may say that the symmetric difference would be 0. Still others may say that it is impossible to determine.

   c. What problems do you note in the above estimates?

      Sample answer: The slope of the secant line is difficult to determine at $x = 0$. However, for some larger *h* values, this slope can be calculated. For example, when $h = 1$, the slope of the symmetric secant near 0 appears to be between –0.204 and 0.204, so 0 is a reasonable guess that students can argue graphically by noting the symmetry in the graph. This is a problem because the symmetric difference quotient will calculate a slope to approximate the derivative at a point even though it does not exist.

Teacher Tip: Because the function **f1**(x) is not defined for $x = 0$, students will not be able to move the x-value to $x = 0$ exactly. However, students should be encouraged to imagine and discuss the location and slope of the secant line for this case. Again, the most important thing to highlight is that the symmetric secant slope exists and will provide an estimate for the derivative of a function at points for which the function is not differentiable.

10. What do the previous two examples caution about when using the symmetric difference quotient to estimate a derivative?

    Sample answer: Because the symmetric difference quotient calculates the slope of a secant using points on either side of (x, **f1**(x)) and not the point itself, this quotient will give an approximation when the function is not defined at the point, not differentiable at the point, or even discontinuous at the point. Therefore, when using the symmetric difference quotient, one must think carefully about the graph of the function.

Extension:

Interestingly, when the function is quadratic, the symmetric secant will give exact values for the derivative of the function, regardless of the choice of h. Students can use the graph on page 5.2 to explore this unique relationship between the symmetric secant approximations and the derivative when **f**(x) is a quadratic function.

You might also begin this exploration by posing one or more of the following questions:

- For what functions, if any, does the symmetric secant provide exact values for the derivative of a function? [It works for polynomial functions of degree 2 or less; are there others?]
- How might you use both graphical and analytic methods to explain why the symmetric secant provides exact answers for the function $f(x) = x^2 - 2x + 3$?

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- How graphically the Symmetric Difference Quotient is represented as the slope of a secant line that can be used to estimate the derivative of a function at a point.
- How numerically the symmetric difference quotient is an average of standard difference quotients, and provides fairly accurate numeric approximations for the derivative of a function without taking a limit.
- Why it is essential to consider the graph of the function when using the symmetric secant to estimate a derivative.

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Note 1

Question 2, *Screen Capture*: Use Screen Capture to display several student screens with different *x*-values chosen. Students can then compare how quickly the secant line approaches the tangent line to (*x*, **f**(*x*)) as *h* decreases for different regions of the graph. Students should visually relate these differences to characteristics of the graph near the point by considering uniform steepness, changes in direction or concavity, relative minimum or maximum points, etc.

Note 2

Question 3, *Screen Capture*: Similar to question 2, this would be a good opportunity to display several screens and compare and discuss the secant and tangent slopes at a number of points along the curve.

Note 3

Questions 2 and 3, *Quick Poll (Multiple Choice or Open Response)*: While investigating question 2, you may wish to have students complete a table of values to explore numerically how the slope of the symmetric secant approaches the slope of the tangent line as the value of *h* decreases and to check that students are able to distinguish between the slope readouts on the screen. This might be done through a Quick Poll or Screen Capture where students explore different points along the curve. The sample table below provides answers for *x* = –5 and *x* = 2.

	Slope of Tangent line	**Slope of Symmetric Secant Line for Given *h* Value**				
		***h* = 2**	***h* = 1**	***h* = .5**	***h* = .25**	***h* = .05**
At *x* = –5	**0.851**	**0.387**	**0.716**	**0.816**	**0.842**	**0.851**
At *x* = 2	**–1.25**	**–.568**	**–1.05**	**–1.2**	**–1.24**	**–1.25**

This table can later be compared to a similar table of values using the standard difference quotients discussed in the next problem. After completing questions 3–7, students can be instructed to go back to the graph on page 1.3 to add the secant lines and measure these slopes for comparison. Given positive values for *h*, table 2 and table 3 below show the slopes using the right and left difference quotients respectively.

	Slope of Tangent line	**Slope of Secant Line from (*x* + *h*, f(*x* + *h*) to (*x*, f(*x*)) for Given *h* Value**				
		***h* = 2**	***h* = 1**	***h* = .5**	***h* = .25**	***h* = .05**
At *x* = –5	**0.851**	**–1.65**	**–0.606**	**0.112**	**0.484**	**0.779**
At *x* = 2	**–1.25**	**–2.5**	**–2.3**	**–1.86**	**–1.57**	**–1.32**

	Slope of Tangent line	**Slope of Secant Line from (*x*, f(*x*) to (*x* – *h*, f(*x* – *h*)) for given *h* Value**				
		***h* = 2**	***h* = 1**	***h* = .5**	***h* = .25**	***h* = .05**
At *x* = –5	**.851**	**2.42**	**2.04**	**1.52**	**1.2**	**0.923**
At *x* = 2	**–1.25**	**1.36**	**0.203**	**–0.529**	**–0.896**	**–1.18**

Students might then use these results for *x* = –5 and *x* = 2 to verify the relationship between the 3 slope values. In particular, the numeric values could be used to demonstrate that the symmetric difference is the average of the right- and left-hand difference quotients by noting examples when *x* = –5 and *h* =.25: (0.484 + 1.2)/2 = 0.842 the symmetric difference quotient.

Note 4
Question 6, *Screen Capture*: This would be a good opportunity to display several screens to compare results for different *x*-values. Given a wider range of numerical examples, students should be able to more quickly conjecture about the relationship between the symmetric secant and the **rdq** and **ldq.**