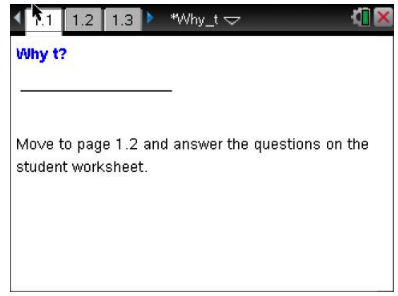
Math Objectives

- Students will recognize that when the population standard deviation is unknown, it must be estimated from the sample in order to calculate a standardized test statistic.
- Students will recognize that when the population standard deviation is known, the standardized test statistic for a sample mean (*z*-score) is unusual only when the sample mean itself is unusual (far from the population mean).
- Students will recognize that when the population standard deviation is unknown, the standardized test statistic for a sample mean (*t*-score) can be unusual for either of two reasons:
  - because the sample mean itself is unusual (far from the population mean), or
  - because the standard deviation of the sample (estimate of population sd) is small.
- Students will recognize that estimating the population standard deviation increases the variability of the sampling distribution of the standardized test statistic for sample means.
- Students will look for and express regularity in repeated reasoning (CCSS Mathematical Practices).

Vocabulary

- hypothesis test
- mean
- population
- sample
- sampling distribution
- standard deviation
- standardized test statistic
- *t*-score
- *z*-score



TI-Nspire™ Technology Skills:
- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:
- Make sure the font size on your TI-Nspire handhelds is set to Medium.

Lesson Files:
Student Activity
Why_*t*_Student.pdf
Why_*t*_Student.doc

TI-Nspire document
Why_*t*.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

About the Lesson

- This lesson involves examining the variability of individual elements and their related standardized test statistics when those elements are drawn randomly from a given normally-distributed population.
- As a result, students will:
  - Confirm that individual elements having large standardized test statistic values (*z*-scores) are exactly the elements that are far from the population mean.

- Observe that large standardized test statistics for sample means (*z*-scores) are directly associated with samples having means far from the population mean.
- Calculate the standardized test statistic for samples taken from a population whose standard deviation is not known (*t*-score) and note that it requires including an estimate for the population standard deviation obtained from the sample itself, namely the sample standard deviation.
- Observe that large values for the standardized test statistic when the population standard deviation is not known (*t*-scores) occur much more frequently than in normal distributions.
- Conclude that the normal distribution no longer describes the sampling distribution of standardized statistics when the population standard deviation is not known. The proper distribution in this case is called the *t* distribution.

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to compare different graphical displays.

Prerequisite Knowledge

- familiarity with the normal distribution, including the Empirical Rule
- the ability to calculate standardized test statistics (*z*-scores)
- familiarity with sampling distributions
- the ability to calculate the mean and standard deviation of the sampling distribution of sample means
- familiarity with hypothesis test logic

Related Activities

- Sampling Distributions
- Standard Error & Sample Means
- Family of *t*-Curves

Discussion Points and Possible Answers

Tech Tip: Page 1.2 gives instructions on how to seed the random number generator on the handheld. Page 1.3 is a *Calculator* page for the seeding process. Ensuring that students carry out this step will prevent students from generating identical data. (Syntax: RandSeed #, where # is a number unique to each student.)

Teacher Tip: The standardized test statistic is called the *z*-score when the statistic is the sample mean and the actual population standard deviation is used in the calculation. It is called a *t*-score when the standard deviation of the population is approximated using the standard deviation of the sample. This lesson examines the sample-to-sample variability in *z*-scores and *t*-scores, focusing on how these sampling distributions differ. The main idea of this activity, as in much of statistical work, is to compare what actually happens as we sample to what we expect to happen so that we can learn from any differences.
This lesson will require students to use results from classmates as well as their own, for comparison and recognition of patterns. The activity might be teacher-led with individual results compiled and shared with the class.

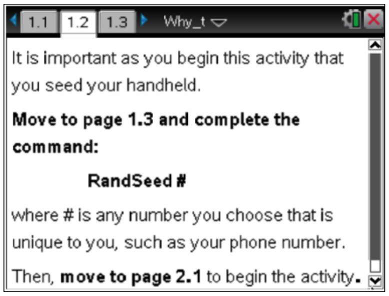
A number of environmental disasters have occurred during the past few decades. Over the years, companies responsible for these problems have begun to display an awareness that they need to take substantial corrective action. After large accidents, part of “making it right” is to set aside a fund to pay business owners for revenue lost due to the accident. One way to determine a fair payment for lost business is to examine revenues for a sample of days during pre-accident times in order to estimate the mean pre-disaster daily revenue. Revenue during the time of reduced business levels following the accident can then be compared to the pre-disaster figure to determine a reasonable payout to the affected business owner.

To study the behavior of a common statistical solution, we will turn to a vastly oversimplified hypothetical example. A large international company, Oops, Inc., has just caused a major local disaster, resulting in loss of business for a small diner in a fishing village. Assume that the diner is open only for lunch, from 11:00 am until 3:00 pm, with daily revenues distributed approximately normally with mean $800 and standard deviation of $150. The distribution of revenue does not vary with the season of the year, etc. The owner of the diner will tell Oops that the mean is $800, but of course, neither the mean nor standard deviation is actually known to Oops. Oops needs to be convinced that the owner's estimate is correct before handing over large sums of money! This is the job of a statistician and why sampling must be done.

In the interest of simplicity, so that the key statistical idea will be clear, this activity will use very small samples. In practice, larger samples would be used. In fact, this activity will show one way in which using larger samples is a good idea!

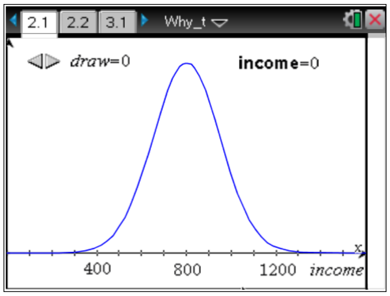
> **Teacher Tip:** Once students have seeded their random number generators, they do not have to do it again unless they have cleared all of the memory. But it is important that this be done if the memory has been cleared or with a new device, as otherwise the "random" numbers will all be the same as those on other similarly cleared devices.

Move to page 1.2.



Note: This activity involves generating a number of random samples from a population. In order to avoid having your results be identical to those for another student in the room, it is necessary to “seed” the random number generator. Read the instructions on Page 1.2 for seeding your random number generator, and carry out that seeding on Page 1.3.

Move to page 2.1.



The graph on Page 2.1 is a portion of the normal distribution curve having mean 800 and standard deviation 150, representing the idealized population of daily revenue from the diner.

1. Suppose you were to select days randomly, one at a time, and record the revenues for the diner for those days. Remembering what you know about normal distributions, write a sentence describing the values you would consider “typical” daily revenue values and values you would consider “unusual.” Explain your reasoning.

   <u>**Sample Answers:**</u> Since the population is normally distributed, the Empirical Rule applies. I would expect daily revenue to be generally within two standard deviations of the mean, i.e., 800 $\pm$ 2(150), so between 500 and 1100. Anything lower than 350 or larger than 1250, i.e., three standard deviations from the mean, would be very unusual, with values between 350 and 500 or between 1100 and 1250 being somewhat unusual.

Tech Tip: Be sure that students understand that the "reset" feature of the left arrow does not just "go back one sample." Rather, it clears the screen for a fresh start on that page.

2. Click on the right arrow on Page 2.1 to generate a random day's revenue. (The left arrow resets the screen.) Is the value you obtain typical or unusual? Compare values across the class to see how many unusual values occurred.

Sample Answers: My value was \$750, which is in the typical range. In my class of 30, though, we had three unusual revenue values.

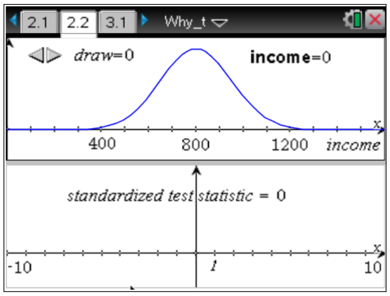
TI-Nspire Navigator Opportunity: ***Screen Capture***
See Note 1 at the end of this lesson.

3. Repeat the process until you get what you think is an unusual revenue value. Leave that value displayed on your screen. How many days (clicks) did you have to examine before obtaining an unusual value? (Note the counter on the arrow.)

Sample Answers: It took me 17 clicks to get a value that I considered unusual.

Teacher Tip: Discuss the relative frequency of "unusual" revenue days within your class. For example, a class of 25 students checking just four samples each results in some 100 samples being examined across the class. Students should agree that the Empirical Rule provides a good description of the observations they as a class have generated.

Move to page 2.2.



The upper panel of the page is exactly the same as page 2.1. The lower panel displays the standardized test statistic (*z*-score) calculated from the randomly-selected day's revenue. Recall that this statistic is calculated using the formula $z = \frac{x-\mu_0}{\sigma}$.

4. a. Record the value of the standardized test statistic for your unusual sample from Question 3 into the table below. How does your standardized test statistic compare to those of your classmates?

Sample Answers: Mine was about the same as others. Some were negative and some were positive, but all were between 2.5 and 3.2 units from 0.

b. Generate four more revenue values, and record the results in the table. Decide whether each revenue value is unusual or typical, and note the associated standardized test statistic value. Write a sentence to summarize how revenue value is related to unusual standardized test statistic values.

Sample Answers: Typical standardized test statistic values (between -2 and 2) come from typical revenue values, while unusual standardized test statistic values come from unusual revenue values.

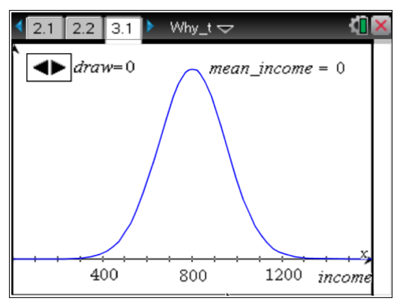
Revenue	Standard Statistic	Unusual or Not
$1240	2.93	Unusual
$620	-1.2	Not unusual
$767	-0.221	Not unusual
$831	0.209	Not unusual
$774	-0.171	Not unusual

Teacher Tip: Discuss class answers to Question 4b carefully. Be sure students see that unusual revenue and unusual *z*-scores go together.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 2 at the end of this lesson.

Move to page 3.1.

The curve on this page is the same as that for the population on previous pages. However, the right arrow now generates a sample of revenues from three randomly selected days and displays the three sample values and the sample mean.



5. Write a sentence to describe the values that represent typical means for samples of size three. What mean values you would consider unusual? Compare your new description of "typical" to the one you wrote in Question 1 and briefly explain any differences.

Sample Answers: Since the sampling distribution of means of samples taken from a normally distributed population is also normally distributed, the Empirical Rule still applies. Typical values of mean revenue (for three days) should be bounded by 800 $\pm$ 2(150/√3), or between about 625 and 975, and "very unusual" means should be lower than about 540 or above about 1060. The new "typical" interval is smaller since the standard deviation for sample means includes division by root n (root 3 in this case).

Teacher Tip: You might need to remind students that the sampling distribution of sample means is less variable than the population from which the samples are drawn (See the Statistics Nspired activity, Standard Error & Sample Means). Be sure all students remember that the standard deviation of the sample means is given by the formula $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$.

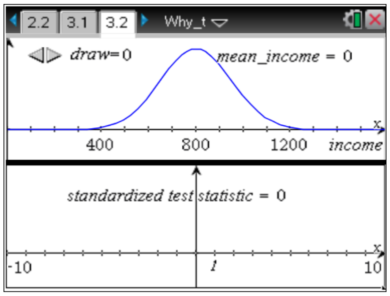
6. Use the arrow to generate samples, observing whether the sample mean appears to be typical or unusual. Stop when you have a sample mean that seems somewhat unusual, and leave that sample displayed on your screen.
   a. How many samples did you have to examine before obtaining an unusual mean?

   Sample Answers: It took me 22 clicks to get a value that I considered unusual.

   b. Did the variability in the sample means that you observed seem consistent with your description in Question 5 of typical means? Explain briefly.

   Sample Answers: Most of the values for the sample means seemed to fall closer to the population mean than single elements did.

Move to page 3.2.



The top work area of this page is exactly the same as Page 3.1. The lower panel displays the standardized test statistic (*z*-score) associated with the sample mean. Recall that the standardized test statistic for sample mean is calculated using the formula $z = \dfrac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}$.

7. a. Record the value of the standardized test statistic for your unusual sample from Question 6 in the table below. How does your standardized test statistic compare to those of your classmates? How do the standardized test statistic values for the class compare to those you observed in Question 4a?

Sample Answers: Mine was about the same as others. Some were negative, some were positive, but all were between 1.9 and 3.2. *Z*-score values now are about the same as back in Question 4.

Mean Revenue	Standard Statistic	Unusual or Not
$628	-1.98	Yes
$974	2.01	Yes
$1040	2.76	Yes
$619	-2.09	Yes

b. Generate three more samples, and add the results to the table on the previous page. In each sample, decide whether the mean revenue value is unusual or typical and note the associated standardized test statistic value. Write a sentence to summarize how samples are related to unusual standardized test statistic values.

Sample Answers: Typical standardized test statistic values (between -2 and 2) come from samples with mean values relatively near the population mean, while unusual standardized test statistic values come from samples with mean values somewhat far from the population mean.

c. Compare your table from Question 4 to this new table. How are decisions about “unusual or not” alike? How are they different?

Sample Answers: The decisions are the same when looking at the standardized statistic, with values much further away from 0 than 2 being considered unusual. However, unusual standardized statistics occur for smaller values of the mean since the standard deviation of means is smaller than that for individual revenues.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 3 at the end of this lesson.

Look back at the formula before Question 7. Notice that the calculation of the standardized test statistic (*z*-score) requires knowing the population mean, the sample mean, the population standard deviation, and the sample size. In reality, the actual population mean and standard deviation of daily revenues are not known to Oops, and Oops does not want to just take the diner owner’s word for it.

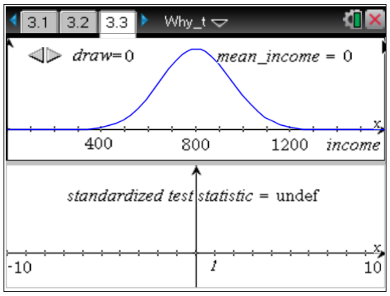
8. The owner's claimed value for the mean becomes the null hypothesis of a hypothesis test, so it is available for use in the *z*-score calculation. However, not knowing the population standard deviation makes the calculation of the z statistic impossible, and makes it difficult to determine if the value is unusual or not.
   a. Explain how you might be able to get a reasonable approximation for the population standard deviation to use in the calculation of the standardized test statistic in this situation.

   Sample Answers: I would do the only thing I can think to do—use the standard deviation of the sample as an approximation for the standard deviation of the population.

   b. Write a sentence predicting how calculating the standardized test statistic using the standard deviation of the sample will affect the set of standardized test statistic values that are judged as unusual.

   Sample Answers: I don't think it will have much effect. Since the samples are selected randomly from the same population, they should be fairly representative of the population, so I think the standardized test statistic will be about the same as before.

Move to page 3.3.



The top work area of the page is again exactly the same as Page 3.1. The bottom work area displays the new standardized test statistic associated with the sample mean, now adjusted to account for the fact that the population standard deviation is unknown. This adjusted standardized test statistic for sample mean is calculated using the formula $z\_adj = \frac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}}$.

9. The last sample you selected in Question 7 is still displayed.
   a. Identify the change you see in the formula for the standardized score. Why is it necessary?

   Sample Answers: In the denominator ***s*** has replaced σ. This is because the value of the population standard deviation is not known.

b. Compare the new standardized test statistic to the one displayed on Page 3.2. Would you call the new value unusual? Explain briefly.

Sample Answers: The values are different, but the new value is still in what I would call the "unusual" range. Changing the denominator of the calculation changed the answer, but not a whole lot.

c. How does your new standardized test statistic compare to those of your classmates? How do the new standardized test statistic values for the class compare to those you observed in Question 7?

Sample Answers: They seem a good bit different now—much more variable. Some were negative and some were positive, and this time some were further out than 3 units. Values now are much different from those in Question 7.

Teacher Tip: Discuss student responses to Question 9c. Be sure that the increased variability—more unusual values, with some of those values being much larger than previously seen—has been noticed.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 4 at the end of this lesson.

10. Generate more samples, trying to get as large (positive or negative) a standardized test statistic as you can, keeping track of the number of clicks it takes to get it. Stop when you think you have an extremely unusual value.

a. How does your new unusual standardized test statistic compare to the unusual values your class got in Question 7? How many clicks did it take you to find it?

Sample Answers: I got a new standardized test statistic value of 14.3! That's way bigger than anything in the entire class when we used the actual population standard deviation in the calculations. And it only took me 10 clicks—bigger and faster!

b. Compare this new "unusual sample" to those you wrote in your table in Question 7. Is this new sample's mean noticeably farther from the population mean than those in your table? Check samples from your classmates, too.

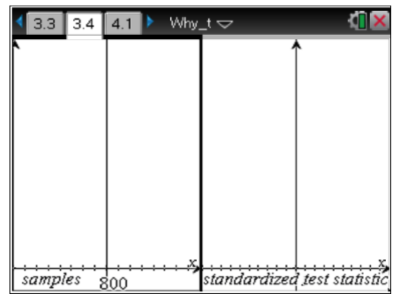
Sample Answers: Even though I got a standardized test statistic of 14.3, the mean of my sample was not that much further from the population mean than I saw in Question 7.

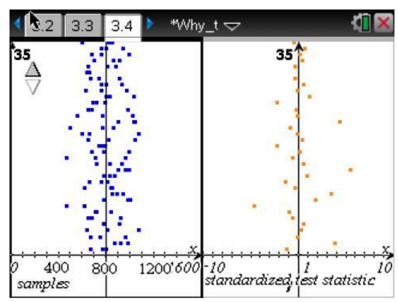
c. Consider both how often “unusual” values of the new statistic appear and how “unusual” they are. Describe how well the Empirical Rule for normal distributions seems to relate to the distribution of these new standardized test statistics.

Sample Answers: The Empirical Rule does not seem appropriate any more. “Big” values seem to occur more often than they should, and they are MUCH larger than we would expect from the Empirical Rule.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 5 at the end of this lesson.

Move to page 3.4.



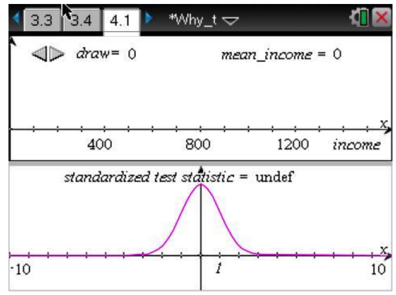


11. The left work area of Page 3.4 shows your most recent samples (without their means), stacked with newer samples at the top and older samples at the bottom. The right panel displays the corresponding values of the adjusted standardized test statistic. Your most recent sample—the unusual one—is displayed at the top in each panel. Study this display and make a conjecture about what causes unusual values of the adjusted standardized test statistic. Remember, the new formula for the standardized test statistic is $z\_adj = \frac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}}$. Use the formula to explain your conjecture.

Sample Answers: My unusual sample has a very small spread, so the value of the standard deviation, ***s***, that I used in my calculations was very small. Dividing by a small number creates a large quotient. The largest values come from having a sample mean that’s pretty far from the population mean and having a tight cluster within the sample.

Teacher Tip: Question 11 is the “punch line” of the activity. Help students discover that the most unusual values of the modified standardize statistic occur for tightly bunched sample values. The formula can then shed light on why this must be true.

Move to page 4.1.



The top work area of Page 4.1 is the same as Page 3.1. The bottom work area of this page displays the standard normal distribution—a normal distribution with mean 0 and standard deviation 1. The right arrow generates and displays a sample of size n = 3, together with its mean, and adds that information into the lower work area. The lower work area then shows the simulated sampling distribution of the standardized test statistic calculated using the standard deviation from the sample as an approximation for the standard deviation of the population as was done for Questions 9 -11.

12. Generate a number of samples, and watch the standardized test statistic value accumulate. Describe how well the standard normal curve predicts this simulated sampling distribution.

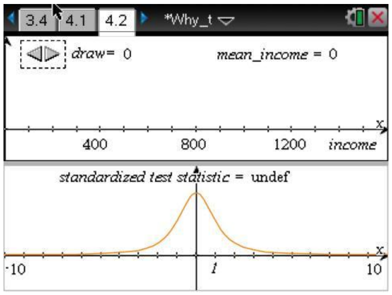
    Sample Answers: The standard normal does a poor job. There are way too many dots falling out in the “tails” of the distribution than the standard normal predicts.

> **Teacher Tip:** Be sure that students notice from their work in Questions 9-12 that the sampling distribution for the standardized test statistic for the sample mean when the population standard deviation is not known is much more spread out than the corresponding statistic when the population standard deviation is known.

Definition: The distribution for this new statistic is called the ***t* distribution.**

> **Teacher Tip:** In fact, “the *t* distribution” is actually a family of distributions. Each particular distribution is determined by the **degrees of freedom**, which is one less than the sample size. So the current activity leads to the *t* distribution having 2 degrees of freedom. For further exploration of this idea, see the TI-Nspired activity called Family of *t*-Curves.

Move to page 4.2.



Page 4.2 is identical to Page 4.1 except that the bottom work area displays the *t* distribution with 2 degrees of freedom.

13. Generate a number of samples, and watch the standardized test statistic value accumulate. Describe how well the *t* distribution predicts this simulated sampling distribution.

    Sample Answers: The *t* distribution does a great job. It seems to conform very well to the actual dots as they accumulate.

14. Suppose that you have been hired as a statistical consultant to Oops as they process claims such as the one from the diner. Each business owner will submit a claim stating their average revenue, and Oops will have to use a sample from past business records to decide whether that claimed average seems consistent with (typical for) that sample. Which distribution (normal or *t*) would you recommend that they use in making that decision. Why?

    Sample Answers: They should use the *t* distribution since they will have the standard deviation only from the sample, not the population.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- When a standardized test statistic calculated using a known population standard deviation (*z*-score) seems unusual, it is always because the sample mean was itself well away from the supposed population mean.
- *Z*-score values larger in magnitude than 3 are very rare when samples come from a normal population and the stated mean is correct.
- "Unusual" values of a standardized test statistic calculated using the sample's standard deviation to approximate the population standard deviation (*t*-score) are frequently much larger in magnitude than those obtained using the actual population standard deviation (*z*-score).
- Any *t*-score values larger in magnitude than 5 are fairly common when using the sample standard deviation and samples of size $n = 3$ from a normal population and the correct mean.
- When an unusual value (say, further out than 5 or so) of the standardized test statistic calculated using the sample standard deviation (*t*-score) occurs, it is because the cases within that sample are very close together and not necessarily because the sample's mean is very far from the population mean! In short, variability within individual samples increases the variability in the sampling distribution of this new statistic.

Assessment

Decide whether you would use a *t* distribution or a normal distribution for the standardized sample statistic for the sample mean in each of the following situations.

1. A random sample of 50 scores from a standardized test with a mean of 70 and a standard deviation of 10.

   Answer: Normal since the population standard deviation is known.

2. A random sample of 30 light bulbs to test the claim that the life expectancy is 240 hours.

   Answer: *t* since the population standard deviation is unknown.

TI-Nspire Navigator

Note 1

Name of Feature: Screen Capture

A Screen Capture can be used to compare sample results from across the class at once to get a quick sense of how often "unusual" revenue days occur.

Note 2

Name of Feature: Screen Capture

A Screen Capture can be used to examine sample results from across the class to look for how "unusual" values of revenue and *z*-score are related.

Note 3

Name of Feature: Screen Capture

A Screen Capture can be used to examine sample results from across the class to look for how "unusual" sample means and *z*-scores are related.

Note 4

Name of Feature: Screen Capture

A Screen Capture can be used to compare sample results from across the class to get a quick sense of how often "unusual" values of the newly modified standardized statistic occur.

Note 5

Name of Feature: Screen Capture

A Screen Capture can be used to compare sample results from across the class to look for how “unusual” *t*-scores are related to their underlying samples.

Note 6

Name of Feature: Quick Poll

Quick Poll can be used to check for student understanding throughout the activity. The suggested assessment questions could be used to check for understanding at the end of the activity.